OCR
Oxford Cambridge and RSA

Thursday 25 May 2023 – Afternoon

A Level Further Mathematics A

Y540/01 Pure Core 1

Time allowed: 1 hour 30 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for A Level Further Mathematics A
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give non-exact numerical answers correct to **3** significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. When a numerical value is needed use $g = 9.8$ unless a different value is specified in the question.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

1 **In this question you must show detailed reasoning.**

Determine the value of $\sum_{r=1}^{50} r^2(16-r)$. **[3]**

2 **In this question you must show detailed reasoning.**

The equation $z^4+4z^3+9z^2+10z+6=0$ has roots α, β, γ and δ.

(a) Show that a quartic equation whose roots are $\alpha+1$, $\beta+1$, $\gamma+1$ and $\delta+1$ is $w^4+3w^2+2=0$. **[3]**

(b) Hence determine the exact roots of the equation $z^4+4z^3+9z^2+10z+6=0$. **[3]**

3 **(a)** Show that $\frac{-3+\sqrt{3}\,\mathrm{i}}{2}=\sqrt{3}\,\mathrm{e}^{\frac{5}{6}\pi\mathrm{i}}$. **[2]**

(b) Hence determine the exact roots of the equation $z^5=\frac{9(-3+\sqrt{3}\,\mathrm{i})}{2}$, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r>0$ and $0\leqslant\theta<2\pi$. **[3]**

4 The transformations T_A and T_B are represented by the matrices **A** and **B** respectively, where

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Describe geometrically the **single** transformation consisting of T_A followed by T_B. **[2]**

(b) By considering the transformation T_A, determine the matrix $\mathbf{A}^{423}$. **[3]**

The transformation T_C is represented by the matrix **C**, where

$$\mathbf{C} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

The region R is defined by the set of points (x, y) satisfying the inequality $x^2 + y^2 \leqslant 36$.

The region R' is defined as the image of R under T_C.

(c) (i) Find the exact area of the region R'. **[2]**

(ii) Sketch the region R', specifying all the points where the boundary of R' intersects the coordinate axes. **[4]**

5 **(a)** Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$. **[2]**

(b) Hence find the general solution of the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = x(4 - 5x)$. **[4]**

6 **In this question you must show detailed reasoning.**

The power output, p watts, of a machine at time t hours after it is switched on can be modelled by the equation $p = 20 - 20\tanh(1.44t)$ for $t \geqslant 0$.

Determine, according to the model, the **mean** power output of the machine over the first half hour after it is switched on. Give your answer correct to **2** decimal places. **[4]**

7 An engineer is modelling the motion of a particle P of mass 0.5 kg in a wind tunnel.

P is modelled as travelling in a straight line. The point O is a fixed point within the wind tunnel. The displacement of P from O at time t seconds is x metres, for $t \geqslant 0$.

You are given that $x \geqslant 0$ for all $t \geqslant 0$ and that P does not reach the end of the wind tunnel.

If $t \geqslant 0$, then P is subject to three forces which are modelled in the following way.

- The first force has a magnitude of $5(t+1)\cosh t$ N and acts in the positive x-direction.
- The second force has a magnitude of $0.5x$ N and acts towards O.
- The third force has a magnitude of $\left|\frac{\mathrm{d}x}{\mathrm{d}t}\right|$ N and acts in the direction of motion of the particle.

(a) The engineer applies the equation "$F = ma$" to the model of the motion of P and derives the following differential equation.

$$5(t+1)\cosh t - 0.5x + \frac{\mathrm{d}x}{\mathrm{d}t} = 0.5\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

(i) Explain the sign of the $\frac{\mathrm{d}x}{\mathrm{d}t}$ term in the engineer's differential equation. **[1]**

When $t = 0$ the displacement of P is 6 m, and it is travelling towards O with a speed of $5\,\mathrm{m\,s^{-1}}$.

(ii) Without attempting to solve the differential equation, find the acceleration of P when $t = 0$. **[2]**

Let the particular solution to the differential equation in part **(a)** be a function f such that $x = \mathrm{f}(t)$ for $t \geqslant 0$.

The particular solution to the differential equation can be expressed as a Maclaurin series.

(b) (i) Show that the Maclaurin series for $\mathrm{f}(t)$ up to and including the term in t is $6 - 5t$. **[1]**

(ii) Use your answer to part **(a)(ii)** to show that the term in t^2 in the Maclaurin series for $\mathrm{f}(t)$ is $-3t^2$. **[1]**

(iii) By differentiating the differential equation in part **(a)** with respect to t, show that the term in t^3 in the Maclaurin series for $\mathrm{f}(t)$ is $0.5t^3$. **[4]**

You are given that the complete Maclaurin series for the function f is valid for all values of $t \geqslant 0$.

After 0.25 seconds P has travelled 1.43 m towards the origin.

(c) (i) By using the Maclaurin series for $\mathrm{f}(t)$ up to and including the term in t^3, evaluate the suitability of the model for determining the displacement of P from O when $t = 0.25$. **[1]**

(ii) Explain why it might not be sensible to use the Maclaurin series for $\mathrm{f}(t)$ up to and including the term in t^3 to evaluate the suitability of the model for determining the displacement of P from O when $t = 10$. **[1]**

8 The points P, Q and R have coordinates (0, 2, 3), (2, 0, 1) and (1, 3, 0) respectively.

The acute angle between the line segments PQ and PR is θ.

(a) Show that $\sin\theta = \frac{2}{11}\sqrt{22}$. **[3]**

The triangle PQR lies in the plane Π.

(b) Determine an equation for Π, giving your answer in the form $ax + by + cz = d$, where a, b, c and d are integers. **[3]**

The point S has coordinates $(5, 3, -1)$.

(c) By finding the shortest distance between S and the plane Π, show that the volume of the tetrahedron $PQRS$ is $\frac{14}{3}$.

[The volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height] **[4]**

The tetrahedron $PQRS$ is transformed to the tetrahedron $P'Q'R'S'$ by a rotation about the y-axis.

The x-coordinate of S' is $2\sqrt{2}$.

(d) By using the matrix for a rotation by angle θ about the y-axis, as given in the Formulae Booklet, determine in exact form the possible coordinates of R'. **[5]**

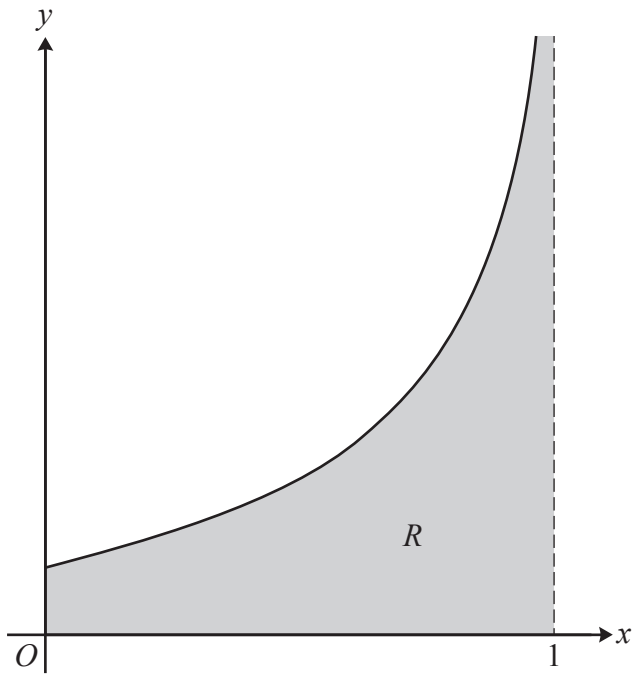
9 In this question you must show detailed reasoning.

(a) Use de Moivre's theorem to determine constants A, B and C such that $\sin^4\theta \equiv A\cos 4\theta + B\cos 2\theta + C$. **[5]**

The function f is defined by

$$f(x) = \sin\left(4\sin^{-1}\left(x^{\frac{1}{5}}\right)\right) - 8\sin\left(2\sin^{-1}\left(x^{\frac{1}{5}}\right)\right) + 12\sin^{-1}\left(x^{\frac{1}{5}}\right), \qquad x \in \mathbb{R},\ 0 \leqslant x < 1.$$

(b) Show that $f'(x) = \dfrac{32}{5\sqrt{1-x^{\frac{2}{5}}}}$. **[6]**



The diagram shows the curve with equation $y = \dfrac{1}{\sqrt{1-x^{\frac{2}{5}}}}$ for $0 \leqslant x < 1$ and the asymptote $x = 1$. The region R is the unbounded region between the curve, the x-axis, the line $x = 0$ and the line $x = 1$.

You are given that the area of R is finite.

(c) Determine the exact area of R. **[3]**

END OF QUESTION PAPER