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A LEVEL

Examiners' report

FURTHER MATHEMATICS A

H245

For first teaching in 2017

Y541/01 Summer 2023 series

Version 1

Contents

Introduction

Our examiners’ reports are produced to offer constructive feedback on candidates’ performance in the examinations. They provide useful guidance for future candidates.

The reports will include a general commentary on candidates’ performance, identify technical aspects examined in the questions and highlight good performance and where performance could be improved. A selection of candidate answers is also provided. The reports will also explain aspects which caused difficulty and why the difficulties arose, whether through a lack of knowledge, poor examination technique, or any other identifiable and explainable reason.

Where overall performance on a question/question part was considered good, with no particular areas to highlight, these questions have not been included in the report.

A full copy of the question paper and the mark scheme can be downloaded from OCR.

Paper Y541/01 series overview

This paper, along with Y540, assesses the compulsory core content of the A Level Further Mathematics qualification. Questions in each paper can assess any part of the core specification. After some years of disruption due to Covid, candidates now generally have a good grasp of the specification. However, missed learning time at lower levels does appear to be impacting on their proficiency with assumed knowledge and skills.

Most candidates appeared to be able to complete the paper in the time available.

Candidates who did well on this paper generally:	Candidates who did less well on this paper generally:
• showed a secure grasp of all standard techniques • communicated well using mathematical language correctly • applied the breadth of their mathematical knowledge to find ways to tackle problems unfamiliar to them • demonstrated the ability to recall and apply the techniques from A Level Maths, especially algebra, trigonometry and calculus, to solve problems on the Further Maths paper.	• had clear gaps in their knowledge of standard techniques • produced unclear or incomplete mathematical arguments • could not find routes into problems unfamiliar to them.

Candidates must give a full analytical solution where the whole question, or specific parts of a question, are preceded by the bold statement request '**In this question you must show detailed reasoning'**. Answers cannot just be given without any justification and any values obtained by a calculator must be supported with sufficient evidence that the mathematical technique is understood. Questions using 'Show that' and 'Determine' also require clear working to be shown.

OCR support

A guide to the command words used in OCR A Level Maths exams can be found on Teach Cambridge.

Question 1 (a), (b), (c) and (d)

1 **(a)** The matrix **P** is given by $\mathbf{P} = \begin{pmatrix} 1 & 0 & -2 & 2 \\ 4 & 2 & -2 & 3 \end{pmatrix}$.

(i) Write down the dimensions of **P**. **[1]**

(ii) Write down the transpose of **P**. **[1]**

(b) The matrices **Q**, **R** and **S** are given by $\mathbf{Q} = (1 \;\; 2)$, $\mathbf{R} = \begin{pmatrix} 3 & -4 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{S} = (3 \;\; -2)$.

Write down the sum of the two of these matrices which are conformable for addition. **[1]**

(c) The dimensions of matrix **A** are 4 by 5. The matrices **A** and **B** are conformable for multiplication so that the matrix **C** = **BA** can be formed. The matrix **C** has 6 rows.

(i) Write down the number of columns that **C** has. **[1]**

(ii) Write down the dimensions of **B**. **[1]**

(iii) Explain whether the matrix **AB** can be formed. **[1]**

(d) Find the value of c for which $\begin{pmatrix} -2 & 3 \\ 6 & 10 \end{pmatrix}\begin{pmatrix} c & 5 \\ 10 & 13 \end{pmatrix} = \begin{pmatrix} c & 5 \\ 10 & 13 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 6 & 10 \end{pmatrix}$. **[2]**

This question, targeting grade E, was mostly well answered as expected. Less successful responses showed a lack of knowledge of the basics of matrix work – dimensions, transpose and conformability. Other less successful responses, in part (c), suggested limited experience of applying problem solving skills on this topic. Almost all candidates, however, were able to solve the problem in part (d). For part (b), candidates should note that a matrix, unlike a coordinate point, doesn't use commas to separate elements.

Question 2 (a)

2 **In this question you must show detailed reasoning.**

(a) Write the complex number $-24+7\mathrm{i}$ in modulus-argument form. **[3]**

The examiners were here assessing understanding of complex numbers. Since this question could have been done entirely by calculator without any real grasp of the required skill set, no marks were available for mere answers without workings.

For part (a), there were four acceptable modulus-argument forms as given in the specification; candidates who gave any other form as their final answer were penalised. A common error was to use the incorrect form $rcos\,\theta + ri\sin\theta$. Another common error was where candidates appeared unaware that there are 2 solutions to $\arctan\frac{-7}{24}$ within the four quadrants, only one of which is in the correct quadrant.

Assessment for learning

When converting cartesian to polar form, most candidates benefit from a sketch of the situation – it helps to identify the range within which theta should lie.

Question 2 (b)

(b) Solve the simultaneous equations given below, giving your answers in cartesian form.

$$\begin{aligned} \mathrm{i}z + 3w &= -7\mathrm{i} \\ -6z + 5\mathrm{i}w &= 3 + 13\mathrm{i} \end{aligned}$$

[4]

Part (b) was generally well answered. Several valid methods were used (see mark scheme), the most common being elimination. It was pleasing to see some candidates using cross-syllabus thinking by taking a matrix approach.

Assessment for learning

A good teaching exercise, to develop problem solving skills, would be to ask students to solve this question using as many different approaches as they can.

A few candidates erroneously assumed that *z* and *w* were real rather than complex variables and so were unable to access any marks.

Exemplar 1

2(b)

$$\begin{pmatrix} i & 3 \\ -6 & 5i \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} -7i \\ 3+13i \end{pmatrix}$$

M

$$M^{-1} = \frac{1}{13}\begin{pmatrix} i & 6 \\ -3 & 5i \end{pmatrix}$$

$$\frac{1}{13}\begin{pmatrix} i & 6 \\ -3 & 5i \end{pmatrix}\begin{pmatrix} -7i \\ 3+13i \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 7+18+96i \\ 21i+15i-65 \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix}$$

$$z = \frac{25+96i}{13} \qquad w = \frac{36i-65}{13}$$

Here is an example of the matrix approach where the candidate was able to pick up method marks by showing that they understood the principal if not how to find an inverse.

Question 3 (a)

3 **(a)** Show that $\frac{\mathrm{d}}{\mathrm{d}u}(\sinh^{-1}u) = \frac{1}{\sqrt{u^2+1}}$. **[2]**

Part (a) caused some problems to many candidates. Among those who used the main approach cited in the mark scheme, only a small number explained why they chose the positive root as required. Those who opted to use the logarithmic form fared slightly better, although many, having used the chain rule, could not simplify their expression.

Question 3 (b)

(b) Find the equation of the normal to the graph of $y = \sinh^{-1} 2x$ at the point where $x = \sqrt{6}$.
Give your answer in the form $y = mx + c$ where m and c are given in exact, non-hyperbolic form. **[4]**

It was pleasing to see part (b) attempted, even when there were clear issues on part (a). This part was generally well answered, but the following errors were seen.

- inability to use the chain rule correctly when differentiating
- omitting the coefficient of *x* (i.e. 2) when finding *y*
- finding the equation of the tangent rather than the normal.

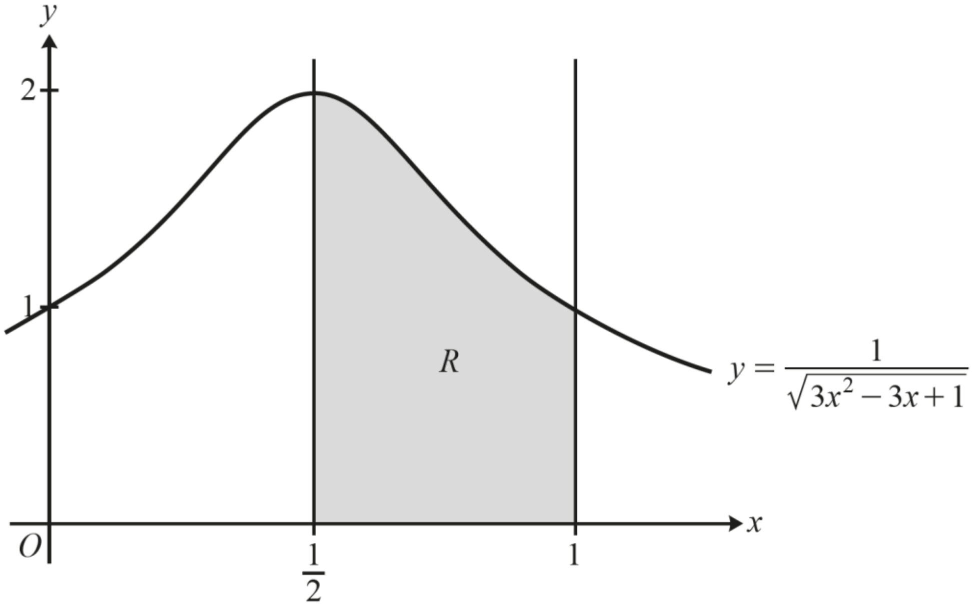
Assessment for learning

It is not uncommon at this level for questions to have a challenging part (a) 'Show that …', with the subsequent part (b) that makes use of that initial result as the starting point. Candidates should make sure they attempt all parts of a question, even if they do not feel confident with the earlier parts.

Question 4

4 **In this question you must show detailed reasoning.**

The region R is bounded by the curve with equation $y = \dfrac{1}{\sqrt{3x^2 - 3x + 1}}$, the x-axis and the lines with equations $x = \dfrac{1}{2}$ and $x = 1$ (see diagram). The units of the axes are cm.



A pendant is to be made out of a precious metal. The shape of the pendant is modelled as the shape formed when R is rotated by 2π radians about the x-axis.

Find the exact value of the volume of precious metal required to make the pendant, according to the model. **[4]**

Most candidates gained the first mark for setting up the appropriate integral and further realised that a completed square form was required. A few used the wrong initial formula, for instance replacing π with $\frac{1}{2}$. There were other candidates who attempted to use partial fractions or just ignored the fact the denominator was not linear and gave the integral as the natural log of the denominator. Some candidates struggled to complete the square correctly.

Those who managed to gain an integral in an appropriate form usually then managed to follow this through to a correct final answer; a few introduced numerical errors usually from confusing $\frac{1}{\frac{1}{\sqrt{12}}}$ with $\frac{1}{\sqrt{12}}$

There were a small number of candidates who, having gained credit for use of the volume formula, clearly used technology to find numerical integrals which gained them no further marks. The question did specify both an exact answer and detailed reasoning.

Question 5 (a)

5 In this question you must show detailed reasoning.

(a) Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, show that $\sinh 2x \equiv 2\sinh x \cosh x$. **[2]**

Candidates appeared better prepared than in previous series with the definition of the hyperbolic trigonometric functions. However, demonstrations of the identity were often not expressed well. Candidates sometimes omitted one end of their argument, showing that exponential expressions were equal, but never explicitly saying that these exponential expressions were equal to the original hyperbolic functions. Other responses omitted equals signs or considered colons or arrows to be equivalent to them.

Exemplar 2

5(a)

$\text{RHS} = 2\sinh x \cosh x$

$= 2 \times \frac{1}{2}\left(e^{x} - e^{-x}\right) \times \frac{1}{2}\left(e^{x} + e^{-x}\right)$

$= \frac{1}{2}\left(e^{2x} + 1 - 1 - e^{-2x}\right)$

$= \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$

$= \sinh 2x = \text{LHS}$

This response follows the ideal joined up right to left (or left to right) line of argument. We progress from one side of the identity to the other via a series of equal expressions. In between we have explicit use of the definitions at both ends of the argument plus an intermediate step to link the two.

Question 5 (b)

(b) Solve the equation $15\sinh x + 16\cosh x - 6\sinh 2x = 20$, giving all your answers in logarithmic form. **[5]**

Fully successful responses mainly saw candidates factorise the equations directly rather than using exponential forms.

The ability to factorise a 2 term quadratic is a skill worth practising due to its frequent problem solving applications. Some candidates attempted this by dividing through by a common factor but then forgot that this common factor also required consideration.

Where candidates used the definitions leading to an exponential quartic, they were then expected to show their process for solving this (using the factor theorem). Those that proceeded by calculator were still able to gain a compensatory mark for the correct solution.

Most candidates who found the 4 possible values of the exponential (from the quartic) recognised that one of them was to be rejected as negative.

Question 6 (a)

6 The equation of the plane Π is $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$.

(a) Find the acute angle between Π and the plane with equation $\mathbf{r}.\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} = 4$. **[4]**

There were many good, clear responses to this question. Marks were lost through calculation errors or by thinking that the required answer needed further manipulation to get to the angle between the two planes.

Question 6 (b)

The point A has coordinates $(9, -7, 20)$.

The point F is the point of intersection between Π and the perpendicular from A to Π.

(b) Determine the coordinates of F. **[4]**

There were half a dozen successful routes through this problem chosen by candidates, although the four given in the mark scheme were the most used, particularly the main method. Apart from those candidates who made no progress, it was usually clear what candidates were trying to do and marks were again lost more through lack of care with manipulation than lack of method.

Question 7 (a)

7 In this question you must show detailed reasoning.

(a) Show that

$$\sum_{r=1}^{n} \frac{5r+6}{r^3+r^2} = \frac{a}{n+1} + b + c\sum_{r=1}^{n}\frac{1}{r^2}$$

where a, b and c are integers whose values are to be determined. **[6]**

Many candidates recognised the need to use partial fractions. However, some omitted the $\frac{A}{r}$ or $\frac{B}{r^2}$ term, and a certain number included both an $\frac{A}{r}$ and a $\frac{Br+C}{r^2}$ term. The latter usually either stated or deduced later on that *A* or *B* was zero. The majority who dealt effectively with the partial fractions went on to attempt simplification of the sum to *n* terms, usually by indicating an explicit list of cancelling fractions, though not all showed sufficient terms at the upper end to qualify for both final marks. A small number of candidates offered the valid algebraic method of writing $\sum_{r=1}^{n}\frac{1}{r+1} = \sum_{r=2}^{n+1}\frac{1}{r} = \sum_{r=2}^{n}\frac{1}{r} + \frac{1}{n+1}$ and $\sum_{r=1}^{n}\frac{1}{r} = \sum_{r=2}^{n}\frac{1}{r} + 1$ and cancelling the common sum.

Question 7 (b)

You are given that $\sum_{r=1}^{\infty}\frac{1}{r^2}$ exists and is equal to $\frac{1}{6}\pi^2$.

(b) Show that $\sum_{r=1}^{\infty}\frac{5r+6}{r^3+r^2}$ exists and is equal to $(\pi-1)(\pi+1)$. **[2]**

Most candidates gained the first mark by indicating that $\frac{1}{n+1}$ tends to zero as $n \to \infty$, although this was frequently expressed poorly; use of incorrect notation, and $\frac{1}{\infty+1}$ or $\frac{1}{\infty}$ were not credited. The second mark was less frequently gained, mainly due to lack of a clear chain of equalities from the sum to the given answer.

Exemplar 3

7(b)

$$\sum_{r=1}^{\infty} \frac{5r+6}{r^3+r^2} = \lim_{N\to\infty} \sum_{r=1}^{N} \frac{5r+6}{r^3+r^2} = \lim_{N\to\infty} 6\sum_{r=1}^{N}\frac{1}{r^2} - 1 + \frac{1}{N+1}$$

$$= 6\lim_{N\to\infty}\sum_{r=1}^{N}\frac{1}{r^2} + \lim_{N\to\infty}(-1) + \lim_{N\to\infty}\left(\frac{1}{N+1}\right)$$

$$= 6\frac{\pi^2}{6} + -1 + 0 \quad \left(\text{line as } N\to\infty,\ \frac{1}{N+1}\to 0\right)$$

$$= \pi^2 - 1 = (\pi-1)(\pi+1)$$

The example given here shows the clarity of argument that the examiners are seeking. The only slip in the candidate's argument is the lack of brackets on their third expression in the first line. Notice the way they equate the question to the answer via an unbroken chain of equalities. Notice also how clear they have made their consideration of each limiting term even when trivial.

Question 8 (a)

8 A surge in the current, I units, through an electrical component at a time, t seconds, is to be modelled. The surge starts when $t = 0$ and there is initially no current through the component. When the current has surged for 1 second it is measured as being 5 units. While the surge is occurring, I is modelled by the following differential equation.

$$(2t-t^2)\frac{dI}{dt} = (2t-t^2)^{\frac{3}{2}} - 2(t-1)I$$

(a) By using an integrating factor show that, according to the model, while the surge is occurring, I is given by $I = (2t-t^2)(\sin^{-1}(t-1)+5)$. **[6]**

Assessment for learning

An ease with the calculus studied in the A Level Maths course is essential for this paper. Centres should give students plenty of opportunity to practise these skills.

Both here, and in Questions 3 (b) and 9 (a) many candidates were let down by poor calculus skills. Here, candidates did not recognise a simple case of a rational expression where the numerator is essentially the derivative of the denominator, leading to either an inability to find the IF (integrating factor) or time wasted on long methods requiring partial fractions.

Many candidates successfully rearranged the equation into the correct form – those that did not were unable to make progress, so this basic step does need emphasising by centres. Most candidates also knew how to use an integrating factor once found.

This is a 'show that' question and since the given answer includes $\sin^{-1}(t-1)$, examiners needed to see a correct completed square version of the expression before integration. Candidates also had to make sure they showed sufficient working in getting $c = 5$.

Question 8 (b)

The surge lasts until there is again no current through the component.

(b) Determine the length of time that the surge lasts according to the model. **[2]**

Most candidates knew they had to put I equal to 0 in the expression from part (a) with answer of $t = 2$ commonly seen. Some, incorrectly, thought that they had solved $\sin^{-1}(t-1) + 5 = 0$ getting $t = 1.96$. The equation, of course, has no solutions since -5 lies outside the range of the arcsin function. If incorrect reasons for this were given, such as claiming that $-1 \leq \sin^{-1} x \leq 1$ then credit could not be given.

Question 8 (c)

(c) Determine, according to the model, the rate of increase of the current at the start of the surge. Give your answer in an exact form. **[3]**

Most candidates who attempted this question used the product rule to differentiate the expression for I The most common mistake was then not recognising that the expression derived had a term that became 0 in the denominator. Very few candidates dealt with this, although, even so, they were able to gain some credit if they found the correct final answer. Those who started with the expression for dI/dt mostly neglected to substitute for I.

Question 9 (a)

9 A function is defined by $y = \mathrm{f}(t)$ where $\mathrm{f}(t) = \ln(1+at)$ and a is a constant.

(a) By considering $\frac{\mathrm{d}y}{\mathrm{d}t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$, $\frac{\mathrm{d}^3y}{\mathrm{d}t^3}$ and $\frac{\mathrm{d}^4y}{\mathrm{d}t^4}$, make a conjecture for a general formula for $\frac{\mathrm{d}^ny}{\mathrm{d}t^n}$ in terms of n and a for any integer $n \geqslant 1$. **[3]**

Quite a few candidates did not correctly differentiate the initial function often getting a numerator of 1 rather than *a*. Further problems arose when candidates considered the higher derivatives.

Those who used the chain rule fared best. Some errors resulted from either treating *a* as a variable, or not simplifying terms along the way, especially when attempting to use quotient rule.

Candidates often produced a conjecture that did not consider the factorial term or ineffectively dealt with the sign of the expression.

Question 9 (b)

(b) Use induction to prove the formula conjectured in part **(a)**. **[4]**

In this question candidates generally demonstrated familiarity with the structure of a proof by mathematical induction. Those that had a conjecture to work with, usually correctly showed that their conjecture worked for $n = 1$. It is important to note that, since the basis case is often quite simple, candidates need to make clear that the conjecture is being used. Candidates usually also attempted to differentiate $\frac{d^ky}{dt^k}$, although some confused this step with that of multiplying by $\frac{dy}{dt}$.

Question 9 (c)

(c) In the case where $\mathrm{f}(t) = \ln(1+2t)$, find the rate at which the 6^{th} derivative of $\mathrm{f}(t)$ is varying when $t = \frac{3}{2}$. **[2]**

Many candidates did not recognise the need to find the seventh derivative here, mistakenly considering only the sixth. Those that did were usually able to reach the correct answer so long as they had been successful previously when differentiating.

Question 10

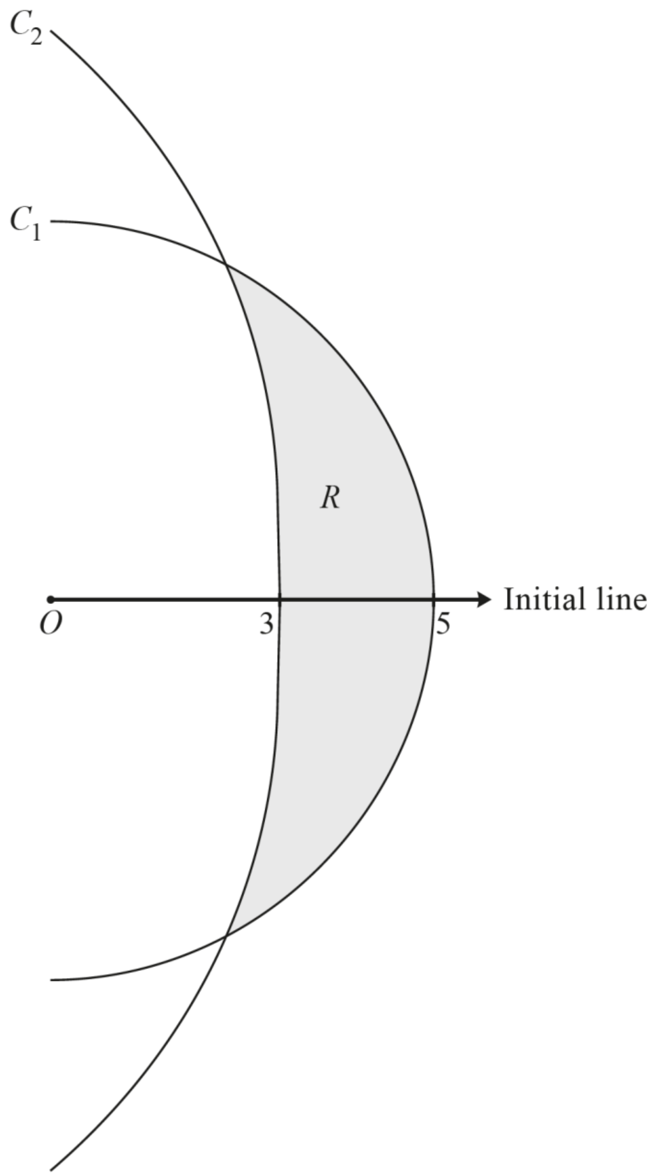
10 In this question you must show detailed reasoning.

A region, R, of the floor of an art gallery is to be painted for the purposes of an art installation. A suitable polar coordinate system is set up on the floor of the gallery with units in metres and radians. R is modelled as being the region enclosed by two curves, C_1 and C_2. The polar equations of C_1 and C_2 are

$$C_1 : r = 5, \qquad -\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$$

$$C_2 : r = 3\cosh\theta, \qquad -\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$$

Both curves are shown in the diagram, with R indicated.



The gallery must buy tins of paint to paint R. Each tin of paint can cover an area of $0.5\,\text{m}^2$.

Determine the smallest number of tins of paint that the gallery must buy in order to be able to paint R completely. **[7]**

A fair number of fully correct solutions were seen to this question, with all necessary detail clearly set out. Some candidates, however, obtained the right answer, but did not show the integration steps, having presumably evaluated the integral on a calculator. As this was a detailed reasoning question such candidates were unable to score more than 5 marks. Other candidates got half the required answer, since they had integrated between 0 and ln3, and forgot to double.

Although most candidates realised that they needed to find the theta values where C_1 and C_2 intersect, some used $\pm\frac{\pi}{2}$ as limits for their integral. Since the final answer to this question is going to be numerical, a numerical theta value at this point is acceptable, though theta = ln3 provides the neatest version.

Some candidates didn't seem to know the correct area formula for polar sectors, omitting the 1/2 or replacing it with π, or using 2r instead of r^2.

A few candidates incorrectly took the area to be given by $\frac{1}{2}\int(5-3\cosh\theta)^2 d\theta$.

Most candidates who avoided the above pitfalls had some idea how to go about integrating $\cosh^2\theta$, either by using a hyperbolic identity or by substituting using exponentials. A number of candidates, however, used an incorrect double angle formula claiming that $\cosh^2\theta = \frac{1}{2}(1-\cosh 2\theta)$.

There were plenty of numerical errors seen, often down to mishandling negatives or doubling/halving erroneously. Candidates would reduce the chance of these happening through clear and systematic working.

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