**OCR**
Oxford Cambridge and RSA

# A Level Further Mathematics A
**Y545 Additional Pure Mathematics**
Sample Question Paper                                    Version 2

# Date – Morning/Afternoon
Time allowed: 1 hour 30 minutes

**You must have**:
Printed Answer Booklet
Formulae A Level Further Mathematics A

**You may use:**
• a scientific or graphical calculator

**INSTRUCTIONS**

• Use black ink. HB pencil may be used for graphs and diagrams only.
• Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
• Answer **all** the questions.
• **Write your answer to each question in the space provided in the Printed Answer Booklet.** Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
• Do **not** write in the bar codes.
• You are permitted to use a scientific or graphical calculator in this paper.
• Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
• The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g$ = 9.8.

**INFORMATION**

• The total number of marks for this paper is **75**.
• The marks for each question are shown in brackets **[ ]**.
• **You are reminded of the need for clear presentation in your answers.**
• The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

Answer **all** the questions.

1    A curve is given by $x = t^2 - 2\ln t$, $y = 4t$ for $t > 0$. When the arc of the curve between the points where $t = 1$ and $t = 4$ is rotated through $2\pi$ radians about the $x$-axis, a surface of revolution is formed with surface area $A$.

Given that $A = k\pi$, where $k$ is an integer,
   • write down an integral which gives $A$ and
   • find the value of $k$.                                                                                                                                                  **[4]**

2    Find the volume of tetrahedron OABC, where O is the origin, A = (2, 3, 1), B = (–4, 2, 5) and C = (1, 4, 4).

   **[3]**

3    Given $z = x\sin y + y\cos x$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} + z = 0$.                                                    **[5]**

4    **(i)** Solve the recurrence relation $u_{n+2} = 4u_{n+1} - 4u_n$ for $n \geq 0$, given that $u_0 = 1$ and $u_1 = 1$.    **[4]**

   **(ii)** Show that each term of the sequence $\{u_n\}$ is an integer.                                                                      **[2]**

5    **In this question you must show detailed reasoning.**

   It is given that $I_n = \displaystyle\int_0^{\pi} \sin^n \theta \, \mathrm{d}\theta$ for $n \geq 0$.

   **(i)** Prove that $I_n = \dfrac{n-1}{n} I_{n-2}$ for $n \geq 2$.                                                                                       **[5]**

   **(ii) (a)** Evaluate $I_1$.                                                                                                                                   **[2]**

   **(b)** Use the reduction formula to determine the exact value of $\displaystyle\int_0^{\pi} \cos^2 \theta \, \sin^5 \theta \, \mathrm{d}\theta$.    **[2]**

6    A surface $S$ has equation $z = \mathrm{f}(x, y)$, where $\mathrm{f}(x, y) = 2x^2 - y^2 + 3xy + 17y$. It is given that $S$ has a single stationary point, $P$.

   **(i) (a)** Determine the coordinates of $P$.                                                                                                   **[5]**

   **(b)** Determine the nature of $P$.                                                                                                                **[3]**

   **(ii)** Find the equation of the tangent plane to $S$ at the point $Q(1, 2, 38)$.                                          **[2]**

**7**    In order to rescue them from extinction, a particular species of ground-nesting birds is introduced into a nature reserve. The number of breeding pairs of these birds in the nature reserve, $t$ years after their introduction, is an integer denoted by $N_t$. The initial number of breeding pairs is given by $N_0$.

An initial discrete population model is proposed for $N_t$.

$$\text{Model I: } N_{t+1} = \tfrac{6}{5} N_t \left(1 - \tfrac{1}{900} N_t\right)$$

**(i) (a)**  For Model I, show that the steady state values of the number of breeding pairs are 0 and 150.   **[3]**

**(b)**  Show that $N_{t+1} - N_t < 150 - N_t$ when $N_t$ lies between 0 and 150.   **[3]**

**(c)**  Hence find the long-term behaviour of the number of breeding pairs of this species of birds in the nature reserve predicted by Model I when $N_0 \in (0, 150)$.   **[2]**

An alternative discrete population model is proposed for $N_t$.

$$\text{Model II: } N_{t+1} = \text{INT}\left(\tfrac{6}{5} N_t \left(1 - \tfrac{1}{900} N_t\right)\right)$$

**(ii) (a)**  Given that $N_0 = 8$, find the value of $N_4$ for each of the two models.   **[2]**

**(b)**  Which of the two models gives values for $N_t$ with the more appropriate level of precision?   **[1]**

**8** The set $X$ consists of all $2 \times 2$ matrices of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, where $x$ and $y$ are real numbers which are not **both** zero.

    **(i) (a)** The matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ are both elements of $X$.

            Show that $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ for some real numbers $p$ and $q$ to be found in terms of $a$, $b$, $c$ and $d$. **[2]**

        **(b)** Prove by contradiction that $p$ and $q$ are not **both** zero. **[5]**

    **(ii)** Prove that $X$, under matrix multiplication, forms a group $G$.
        [You may use the result that matrix multiplication is associative.] **[4]**

  **(iii)** Determine a subgroup of $G$ of order 17. **[2]**

**9**    **(i) (a)** Prove that $p \equiv \pm 1 \,(\text{mod}\,6)$ for all primes $p > 3$. **[2]**

        **(b)** Hence or otherwise prove that $p^2 - 1 \equiv 0 \,(\text{mod}\,24)$ for all primes $p > 3$. **[3]**

    **(ii)** Given that $p$ is an odd prime, determine the residue of $2^{p^2-1}$ modulo $p$. **[4]**

  **(iii)** Let $p$ and $q$ be distinct primes greater than 3. Prove that $p^{q-1} + q^{p-1} \equiv 1 \,(\text{mod}\,pq)$. **[5]**

**END OF QUESTION PAPER**