Paper 4A: Further Pure Mathematics 2 Mark Scheme

Question	Scheme	Marks	AOs
1(i)	$602 = 3\times161+119$	M1	1.1b
	$161 = 119 + 42,\ 119 = 2\times 42 + 35$	M1	1.1b
	$42 = 35 + 7,\ 35 = 5\times 7,\ \text{hcf} = 7$	A1	1.1b
		(3)	
(ii)	Number of codes under old system = $5\times4\times4\times3\times2\ (= 480)$	B1	3.1b
	Number of codes under new system = $4\times3\times7\times6\times5\ (= 2520)$	B1	3.1b
	Subtracts first answer from second	M1	1.1b
	Increase in number of codes is 2040	A1	1.1b
		(4)	
			(7 marks)

Notes:

(i)

M1: Attempts Euclid's algorithm – (there may be an arithmetic slip finding 119)

M1: Uses Euclid's algorithm a further two times with 161 and "their 119" and then with "their 119" and "their 42"

A1: This should be accurate with all the steps shown

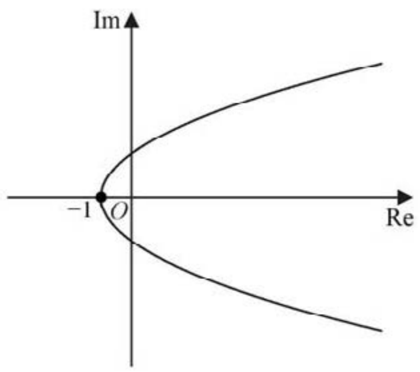
(ii)

B1: Correctly interprets the problem and uses the five odd digits and four even digits to form a correct product

B1: Interprets the new situation using the four even digits, then the seven digits which have not been used, to form a correct product

M1: Subtracts one answer from the other

A1: Correct answer

Question	Scheme	Marks	AOs
2(a)	Let $z = x + \mathrm{i}$	M1	2.1
	$w = (x+\mathrm{i})^2 = (x^2 - 1) + 2x\mathrm{i}$	A1	1.1b
	Let $w = u + \mathrm{i}v$, then $u = (x^2 - 1)$ and $v = 2x$	M1	2.1
	$\Rightarrow v^2 = 4(u+1)$, which therefore represents a parabola	A1ft	2.2a
		(4)	
(b)	 M1: Sketches a parabola with symmetry about the real axis A1: Accurate sketch	M1 A1	1.1b 1.1b
		(2)	
			(6 marks)

Notes:

(a)
M1: Translates the information that Im(z) = 1 into a cartesian form; e.g. $z = x + \mathrm{i}$
A1: Obtains a correct expression for w
M1: Separates the real and imaginary parts and equates to u and v respectively
A1ft: Obtains a quadratic equation and states that their quadratic equation represents a parabola

(b)
M1: Sketches a parabola with symmetry about the real axis
A1: Accurate sketch

Question	Scheme		Marks	AOs
3(a)	Finds the characteristic equation $(2-\lambda)^2(4-\lambda)-(4-\lambda)=0$		M1	2.1
	So $(4-\lambda)(\lambda^2-4\lambda+3)=0$ so $\lambda=4$ *		A1*	2.2a
	Solves quadratic equation to give		M1	1.1b
	$\lambda=1$ and $\lambda=3$		A1	1.1b
			(4)	
(b)	Uses a correct method to find an eigenvector		M1	1.1b
	Obtains a vector parallel to one of $\begin{pmatrix}0\\0\\1\end{pmatrix}$ or $\begin{pmatrix}1\\1\\1\end{pmatrix}$ or $\begin{pmatrix}3\\-3\\1\end{pmatrix}$		A1	1.1b
	Obtains two correct vectors		A1	1.1b
	Obtains all three correct vectors		A1	1.1b
			(4)	
(c)	Uses their three vectors to form a matrix		M1	1.2
	$\begin{pmatrix}0&1&3\\0&1&-3\\1&1&1\end{pmatrix}$	**or** other correct answer with columns in a different order	A1	1.1b
			(2)	
			(10 marks)	

Notes:

(a)

M1: Attempts to find the characteristic equation (there may be one slip)

A1*: Deduces that $\lambda=4$ is a solution by the method shown or by checking that $\lambda=4$ satisfies the characteristic equation

M1: Solves their quadratic equation

A1: Obtains the two correct answers as shown above

(b)

M1: Uses a correct method to find an eigenvector

A1: Obtains one correct vector (may be a multiple of the given vectors)

A1: Obtains two correct vectors (may be multiples of the given vectors)

A1: Obtains all three correct vectors (may be multiples of the given vectors)

(c)

M1: Forms a matrix with their vectors as columns

A1: $\begin{pmatrix}0&1&3\\0&1&-3\\1&1&1\end{pmatrix}$ **or** $\begin{pmatrix}1&0&3\\1&0&-3\\1&1&1\end{pmatrix}$ **or** $\begin{pmatrix}3&1&0\\-3&1&0\\1&1&1\end{pmatrix}$ **or** other correct alternative

Question	Scheme	Marks	AOs
4(i)	If we assume $ab = ba$; as $a^2b = ba$ then $ab = a^2b$	M1	2.1
	So $a^{-1}abb^{-1} = a^{-1}a^2bb^{-1}$	M1	2.1
	So $e = a$	A1	2.2a
	But this is a contradiction, as the elements e and a are distinct so $ab \neq ba$	A1	2.4
		(4)	
(ii)(a)	2 has order 4 and 4 has order 2	M1	1.1b
	7, 8 and 13 have order 4	A1	1.1b
	11 and 14 have order 2 and 1 has order 1	A1	1.1b
		(3)	
(ii)(b)	Finds the subgroup {1, 2, 4, 8} or the subgroup {1, 7, 4, 13}	M1	1.1b
	Finds both and refers to them as cyclic groups, or gives generator 2 and generator 7	A1	2.4
	Finds {1, 4, 11, 14}	B1	2.2a
	States each element has order 2 or refers to it as Klein Group	B1	2.5
		(4)	
(ii)(c)	*J* has an element of order 8, (*H* does not) or *J* is a cyclic group (*H* is not) or other valid reason	M1	2.4
	They are not isomorphic	A1	2.2a
		(2)	
			(13 marks)

Question 4 notes:	
(i)	
M1:	Proof begins with assumption that $ab = ba$ and deduces that this implies $ab = a^2b$
M1:	A correct proof with working shown follows, and may be done in two stages
A1:	Concludes that assumption implies that $e = a$
A1:	Explains clearly that this is a contradiction, as the elements *e* and *a* are distinct so $ab \neq ba$
(ii)(a)	
M1:	Obtains two correct orders (usually the two in the scheme)
A1:	Finds another three correctly
A1:	Finds the final three so that all eight are correct
(ii)(b)	
M1:	Finds one of the cyclic subgroups
A1:	Finds both subgroups and explains that they are cyclic groups, or gives generators 2 and 7
B1:	Finds the non cyclic group
B1:	Uses correct terms that each element has order 2 or refers to it as Klein Group
(ii)(c)	
M1:	Clearly explains how *J* differs from *H*
A1:	Correct deduction

Question	Scheme	Marks	AOs
5(a)	$\frac{dy}{dx} = -\sinh 2x$	B1	2.1
	So $S = \int \sqrt{1+\sinh^2 2x}\,dx$	M1	2.1
	$\therefore s = \int \cosh 2x dx$	A1	1.1b
	$= \left[\frac{1}{2}\sinh 2x\right]_{-\ln a}^{\ln a}$ or $\left[\sinh 2x\right]_0^{\ln a}$	M1	2.1
	$= \sinh 2\ln a = \frac{1}{2}[e^{2\ln a} - e^{-2\ln a}] = \frac{1}{2}\left(a^2 - \frac{1}{a^2}\right)$ (so k = ½)	A1	1.1b
		(5)	
(b)	$\frac{1}{2}\left(a^2 - \frac{1}{a^2}\right) = 2$ so $a^4 - 4a^2 - 1 = 0$	M1	1.1b
	$a^2 = 2+\sqrt{5}$ (and a = 2.06 (approx.))	M1	1.1b
	When x = ln$a$, y = 0 so $A = \frac{1}{2}\cosh(2\ln a)$	M1	3.4
	Height = A – 0.5 = awrt 0.62m	A1	1.1b
		(4)	
(c)	The width of the base = 2ln$a$ = 1.4m	B1	3.4
		(1)	
(d)	A parabola of the form $y = 0.62 - 1.19x^2$, or other symmetric curve with its equation e.g. 0.62cos(2.2x)	M1A1	3.3 3.3
		(2)	
			(12 marks)

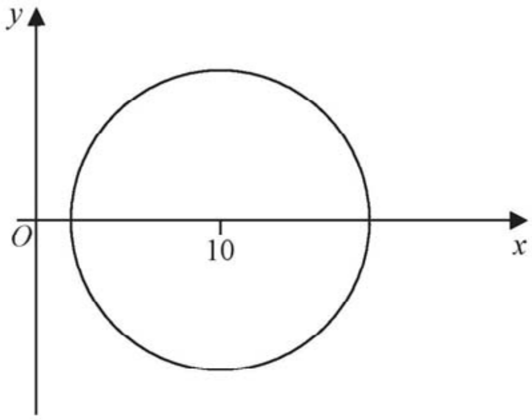
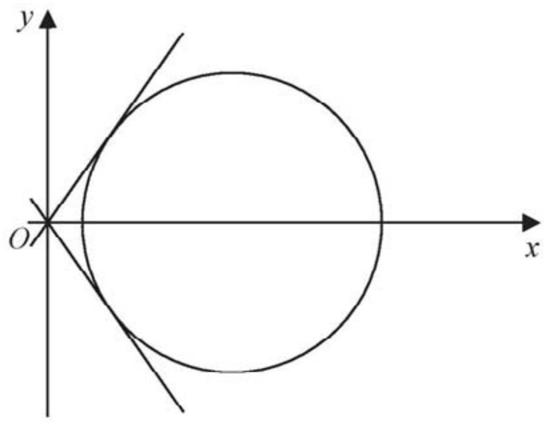
Notes:

(a)
B1: Starts explanation by finding the correct derivative
M1: Uses their derivative in the formula for arc length
A1: Uses suitable identity to simplify the integrand and to obtain the expression in scheme
M1: Integrates and uses appropriate limits to find the required arc length
A1: Uses the definition of sinh to complete the proof and identifies the value for k

(b)
M1: Uses the formula obtained from the model and the length of the arch to create a quartic equation
M1: Continues to use this model to obtain a quadratic and to obtain values for a
M1: Attempts to find a value for A in order to find h
A1: Finds a value for the height correct to 2sf (or accept exact answer)

(c)
B1: Finds width to 2 sf i.e. 1.4m

(d)
M1: Chooses or describes an even function with maximum point on the y axis
A1: Gives suitable equation passing through (0, 0.62) and (0.7, 0) and (- 0.7, 0)

Question	Scheme	Marks	AOs
6(a)	$(x+6)^2 + y^2 = 4[(x-6)^2 + y^2]$	M1	2.1
	$x^2 + y^2 - 20x + 36 = 0$ which is the equation of a circle*	A1*	2.2a
		(2)	
(b)		M1 A1	1.1b 1.1b
		(2)	
(c)	Let $a = c + \mathrm{i}d$ and $a^* = c - \mathrm{i}d$ then $(c + \mathrm{i}d)(x - \mathrm{i}y) + (c - \mathrm{i}d)(x + \mathrm{i}y) = 0$	M1	3.1a
	So $y = -\dfrac{c}{d}x$	A1	1.1b
	 The gradients of the tangents (from geometry) are $\pm\dfrac{4}{3}$	B1	3.1a
	So $-\dfrac{c}{d} = \pm\dfrac{4}{3}$ and $\dfrac{d}{c} = \mp\dfrac{3}{4}$	M1	3.1a
	So $\tan\theta = \pm\dfrac{3}{4}$	A1	1.1b
		(5)	

Question 6 notes:
(a) **M1:** Obtains an equation in terms of x and y using the given information **A1*:** Expands and simplifies the algebra, collecting terms and obtains a circle equation correctly, deducing that this is a circle
(b) **M1:** Draws a circle with centre at (10, 0) **A1:** (Radius is 8) so circle does not cross the y axis
(c) **M1:** Attempts to convert line equation into a cartesian form **A1:** Obtains a simplified line equation **B1:** Uses geometry to deduce the gradients of the tangents **M1:** Understands the connection between arg a and the gradient of the tangents and uses this connection **A1:** Correct answers

Question	Scheme	Marks	AOs
7(a)	$I_n = \int_0^{\frac{\pi}{2}} \sin x \sin^{n-1} x \, dx$	M1	2.1
	$= \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} - (-)\int_0^{\frac{\pi}{2}} \cos^2 x(n-1)\sin^{n-2} x \, dx$	A1	1.1b
	Obtains $= 0 - (-)\int_0^{\frac{\pi}{2}} (1-\sin^2 x)(n-1)\sin^{n-2} x \, dx$	M1	1.1b
	So $I_n = (n-1)I_{n-2} - (n-1)I_n$ and hence $nI_n = (n-1)I_{n-2}$ *	A1*	2.1
		(4)	
(b)	uses $I_n = \frac{(n-1)}{n} I_{n-2}$ to give $I_{10} = \frac{9}{10} I_8$ or $I_2 = \frac{1}{2} I_0$	M1	3.1b
	So $I_{10} = \frac{9\times 7\times 5\times 3\times 1}{10\times 8\times 6\times 4\times 2} I_0$ x	M1	2.1
	$I_0 = \frac{\pi}{2}$	B1	1.1b
	Required area is $2(I_2 - I_{10}) =$ **or** $8\times\frac{1}{4}(I_2 - I_{10}) =$	M1	3.1b
	$= 2\left(\frac{\pi}{4} - \frac{63\pi}{512}\right) = \frac{65\pi}{256} \text{m}^2$	A1	1.1b
		(5)	
			(9 marks)

Notes:

(a)
M1: Splits the integrand into the product shown and begins process of integration by parts (there may be sign errors)
A1: Correct work
M1: Uses limits on the first term and expresses $\cos^2$ term in terms of $\sin^2$
A1*: Completes the proof collecting I_n terms correctly with all stages shown

(b)
M1: Attempts to find I_{10} and/or I_2
M1: Finds I_{10} in terms of I_0
B1: Finds I_0 correctly
M1: States the expression needed to find the required area
A1: Completes the calculation to give this exact answer

Question	Scheme	Marks	AOs
8(a)	$u_1 = 1$ as there is only one way to go up one step	B1	2.4
	$u_2 = 2$ as there are two ways: one step then one step or two steps	B1	2.4
	If first move is one step then can climb the other $(n-1)$ steps in u_{n-1} ways. If first move is two steps can climb the other $(n-2)$ steps in u_{n-2} ways so $u_n = u_{n-1} + u_{n-2}$	B1	2.4
		(3)	
(b)	Sequence begins 1, 2, 3, 5, 8, 13, 21, 34,… so 34 ways of climbing 8 steps	B1	1.1b
		(1)	
(c)	To find general term use $u_n = u_{n-1} + u_{n-2}$ gives $\lambda^2 = \lambda + 1$	M1	2.1
	This has roots $\frac{1 \pm \sqrt{5}}{2}$	A1	1.1b
	So general form is $A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$	M1	2.2a
	Uses initial conditions to find *A* and *B* reaching two equations in *A* and *B*	M1	1.1b
	Obtains $A = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)$ and $B = -\left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)$ and so when $n = 400$ obtains $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{401} - \left(\frac{1-\sqrt{5}}{2}\right)^{401}\right]$ *	A1*	1.1b
		(5)	
		(9 marks)	

Notes:

(a)

B1: Need to see explanation for $u_1 = 1$

B1: Need to see explanation for $u_2 = 2$ with the two ways spelled out

B1: Need to see the first move can be one step or can be two steps and clear explanation of the iterative expression as in the scheme

(b)

B1: The answer is enough for this mark

(c)

M1: Obtains this characteristic equation

A1: Solves quadratic – giving exact answers

M1: Obtains a general form

M1: Use initial conditions to obtains two equations which should be $A\left(1+\sqrt{5}\right) + B\left(1-\sqrt{5}\right) = 2$ o.e. and $A\left(3+\sqrt{5}\right) + B\left(3-\sqrt{5}\right) = 4$ but allow slips here

A1*: Must see exact correct values for *A* and *B* and conclusion given for $n = 400$