GCE A LEVEL MARKING SCHEME

SUMMER 2023

A LEVEL
FURTHER MATHEMATICS
UNIT 4 FURTHER PURE MATHEMATICS B
1305U40-1

INTRODUCTION

This marking scheme was used by WJEC for the 2023 examination. It was finalised after detailed discussion at examiners' conferences by all the examiners involved in the assessment. The conference was held shortly after the paper was taken so that reference could be made to the full range of candidates' responses, with photocopied scripts forming the basis of discussion. The aim of the conference was to ensure that the marking scheme was interpreted and applied in the same way by all examiners.

It is hoped that this information will be of assistance to centres but it is recognised at the same time that, without the benefit of participation in the examiners' conference, teachers may have different views on certain matters of detail or interpretation.

WJEC regrets that it cannot enter into any discussion or correspondence about this marking scheme.

WJEC GCE A LEVEL FURTHER MATHEMATICS

UNIT 4 FURTHER PURE MATHEMATICS B

SUMMER 2023 MARK SCHEME

Qu.	Solution	Mark	Notes
1. a)	Domain of fg: $(0,\infty)$ Range of fg: $[1,\infty)$	B1 B1 **(2)**	
b)	$\cosh(x^2-1)=3$ $x^2-1=\cosh^{-1}3$ $x^2=1+\cosh^{-1}3$ $x=\pm1.662$ $x=1.662$ since $x=-1.662$ not in domain	B1 M1 A1 **(3)** **[5]**	A0 for $x=\pm1.662$ as final answer
2. a)	det **A** = $\pm\lambda(2\lambda-16)\pm1(-\lambda+24)\pm14(-2+6)$ det **A** = $\lambda(2\lambda-16)-1(-\lambda+24)+14(-2+6)$ det **A** = $2\lambda^2-15\lambda+32$	M1 A1 A1 **(3)**	Or equivalent rows/columns Fully simplified
b)	**A** is singular if det **A** = 0 METHOD 1: $2\lambda^2-15\lambda+32=2\left(\lambda-\frac{15}{4}\right)^2+\frac{31}{8}$ $2\left(\lambda-\frac{15}{4}\right)^2+\frac{31}{8}>0$ for all values of λ. Therefore **A** is non-singular. METHOD 2: When $2\lambda^2-15\lambda+32=0$, $b^2-4ac=225-256=-31$ As $-31<0$ there are no real roots. Therefore **A** is non-singular. METHOD 3: When $2\lambda^2-15\lambda+32=0$, $\lambda=\frac{15\pm\sqrt{225-256}}{4}$ $\lambda=\frac{15}{4}\pm\frac{\sqrt{31}}{4}\mathrm{i}$ Therefore, there are no real roots. Therefore **A** is non-singular.	B1 M1 A1 B1 (M1) (A1) (B1) (M1) (A1) (B1) **(4)** **[7]**	si FT from (a) Accept $2\left(\lambda-\frac{15}{4}\right)^2+\frac{31}{8}\neq0$ for real values of λ

Qu.	Solution	Mark	Notes
3.a)	$z^n = \cos n\theta + \mathrm{i}\sin n\theta$ $\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ $= \cos(n\theta) - \mathrm{i}\sin(n\theta)$ $\therefore z^n + \frac{1}{z^n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos(n\theta) - \mathrm{i}\sin(n\theta)$ $= 2\cos n\theta$	M1 A1 A1 **(3)**	1 use of de Moivre's theorem Both z^n and z^{-n} Convincing
b)	$\left(z + \frac{1}{z}\right)^6$ $= z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}$ $= (z^6 + z^{-6}) + (6z^4 + 6z^{-4}) + (15z^2 + 15z^{-2}) + 20$ $= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ $\therefore (2\cos\theta)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ $(2\cos\theta)^6 = 64\cos^6\theta$ $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$	M1 A2 m1 A1 A1 **(6)** **[9]**	Attempt to expand A1 at least 4 terms correct Condone unsimplified z terms si at least 2 pairs correct cao FT cao above
4.	METHOD 1: $\begin{pmatrix} 4 & -2 & 3 \\ 2 & -3 & 8 \\ 2 & 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix}$ By row operations: eg. $\begin{pmatrix} 1 & -3/2 & 4 \\ 0 & 4 & -13 \\ 0 & -1 & 17 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/2 \\ 10 \\ -19 \end{pmatrix}$ Then $\begin{pmatrix} 1 & -3/2 & 4 \\ 0 & -1 & 17 \\ 0 & 0 & 55 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/2 \\ -19 \\ -66 \end{pmatrix}$ Solving, $z = \frac{-6}{5}$ $y = \frac{-7}{5}$ $x = \frac{11}{5}$	M1 A1 A1 m1 A1	Attempt to reduce to echelon form (formal matrix notation not required) Reduction to $\begin{pmatrix} k & & \\ 0 & & \\ 0 & & \end{pmatrix}$ Reduction to $\begin{pmatrix} k & & \\ 0 & & \\ 0 & 0 & \end{pmatrix}$ cao

Qu.	Solution	Mark	Notes
	METHOD 2:		
	Rearranging $x = -2y + \frac{1}{2}z$ and Substituting:	(M1)	
	1st equation: $-10y + 5z = 8$	(A1)	Must be simplified
	2nd equation: $-7y + 9z = -1$	(A1)	Must be simplified
	Solving,	(m1)	
	$z = \frac{-6}{5}$ $\quad y = \frac{-7}{5}$ $\quad x = \frac{11}{5}$	(A1)	cao
	METHOD 3:		
	$\begin{pmatrix} 4 & -2 & 3 \\ 2 & -3 & 8 \\ 2 & 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix}$	(M1)	Attempt to reduce to echelon form (formal matrix notation not required)
	By row operations: eg. $\begin{pmatrix} 1 & -3/2 & 4 \\ 0 & 4 & -13 \\ 0 & -1 & 17 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1/2 \\ 10 \\ -19 \end{pmatrix}$	(A1)	Reduction to $\begin{pmatrix} k & & \\ 0 & & \\ 0 & & \end{pmatrix}$
	Then writing a pair of simultaneous equations in two variables e.g. $4y - 13z = 10$, $-y + 17z = -19$	(A1)	Must be simplified
	Solving,	(m1)	
	$z = \frac{-6}{5}$ $\quad y = \frac{-7}{5}$ $\quad x = \frac{11}{5}$	(A1)	cao
	METHOD 4:		
	Attempt to calculate inverse matrix:	(M1)	
	$\frac{1}{-110}\begin{pmatrix} -29 & 10 & -7 \\ 18 & -10 & -26 \\ 14 & -20 & -8 \end{pmatrix}$	(A1)	At least 5 entries correct in correct position (det not required)
		(A1)	Fully correct (including det)
	Multiplying matrices $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-110}\begin{pmatrix} -29 & 10 & -7 \\ 18 & -10 & -26 \\ 14 & -20 & -8 \end{pmatrix}\begin{pmatrix} 8 \\ -1 \\ 0 \end{pmatrix}$	(m1)	
	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{11}{5} \\ -\frac{7}{5} \\ -\frac{6}{5} \end{pmatrix}$	(A1)	
		[5]	M0 for answers without supporting working

Qu.	Solution	Mark	Notes
5. a)	METHOD 1: $\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!}$ $\sin 2x = 2x - \frac{8}{6}x^3 + \frac{32}{120}x^5 + \cdots$ $\sin 2x = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + \cdots$ METHOD 2: $\sin 2x = \sin 0 + 2x\cos 0 + \frac{(2x)^2}{2!}(-\sin 0)$ $+ \frac{(2x)^3}{3!}(-\cos 0) + \frac{(2x)^4}{4!}\sin 0 + \frac{(2x)^5}{5!}\cos 0$ $\sin 2x = 2x - \frac{8}{6}x^3 + \frac{32}{120}x^5 + \cdots$ $\sin 2x = 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + \cdots$	M1 A1 (M1) (A1) **(2)**	Use of formula booklet expansion of $\sin x$ Use of formula booklet expansion of $f(x)$ with $\sin x$ and $2x$
b)	Differentiating, $2\cos 2x = 2 - 4x^2 + \frac{4}{3}x^4$ $\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4$ $\cos 2x = 2\cos^2 x - 1$ Therefore, $1 - 2x^2 + \frac{2}{3}x^4 = 2\cos^2 x - 1$ $2\cos^2 x = 2 - 2x^2 + \frac{2}{3}x^4$ $\cos^2 x = 1 - x^2 + \frac{1}{3}x^4$	M1 A1 m1 A1 A1 **(5)** **[7]**	FT (a) oe cao

Qu.	Solution	Mark	Notes
6. a)	METHOD 1: $\tan\theta = \tan\left(\frac{\theta}{2}+\frac{\theta}{2}\right)$ $\tan\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = \frac{\tan\left(\frac{\theta}{2}\right)+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\theta}{2}\right)}$ $\tan\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = \frac{t+t}{1-t\times t}$ $\tan\theta = \frac{2t}{1-t^2}$	E1	Convincing
	METHOD 2: For $t = \tan\left(\frac{\theta}{2}\right)$, $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ Therefore, $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}} = \frac{2t}{1-t^2}$	(E1) **(1)**	Convincing
b)	$x = r\cos(\theta) = \cos\left(\frac{\theta}{2}\right)\cos\theta$	M1	
	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\cos\left(\frac{\theta}{2}\right)\sin\theta - \frac{1}{2}\cos\theta\sin\left(\frac{\theta}{2}\right)$	m1	Use of product rule
	When perpendicular to initial line: $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\cos\left(\frac{\theta}{2}\right)\sin\theta - \frac{1}{2}\cos\theta\sin\left(\frac{\theta}{2}\right) = 0$	A1	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$
	$-\cos\left(\frac{\theta}{2}\right)\sin\theta = \frac{1}{2}\cos\theta\sin\left(\frac{\theta}{2}\right)$ $\tan(\theta) = -\frac{1}{2}\tan\left(\frac{\theta}{2}\right)$	A1 **(4)**	Convincing

Qu.	Solution	Mark	Notes
6. c)	$\frac{2t}{1-t^2} = -\frac{1}{2}t$	M1	Use of compound-angle Accept use of $\tan\left(\frac{\theta}{2}\right)$ in place of t throughout
	$2t = -\frac{1}{2}t + \frac{1}{2}t^3$		
	$4t = -t + t^3$		
	$t^3 - 5t = 0$	A1	
	$t(t^2 - 5) = 0$		
	$t = 0, \quad t = \pm\sqrt{5}$	A1	For at least 2 solutions
	$\tan\left(\frac{\theta}{2}\right) = 0$ or $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{5}$		
	$\frac{\theta}{2} = 0 \quad \frac{\theta}{2} = 1.15 \ldots \quad \frac{\theta}{2} = -1.15 \ldots$	A1	Ignore angles outside range
	$\theta = 0 \quad r = 1$ $\theta = 2.30 \quad r = 0.408$ $\theta = -2.30 \quad r = 0.408$	B2	FT their $\frac{\theta}{2}$ values provided $r > 0$ B2 All correct, no additional sets B1 for 2 correct sets of coordinates
	(1, 0), (0.408, 2.30), (0.408, -2.30)	**(6)**	
d)	Area = $\frac{1}{2}\int_0^{\pi} \cos^2\left(\frac{\theta}{2}\right) \mathrm{d}\theta$	M1	
	$= \frac{1}{2}\int_0^{\pi} \frac{\cos\theta + 1}{2} \mathrm{d}\theta$	M1 A1	Use of double-angle formula All correct, including limits
	$= \frac{1}{4}\int_0^{\pi} (\cos\theta + 1)\, \mathrm{d}\theta$		
	$= \frac{1}{4}[\sin\theta + \theta]_0^{\pi}$	A1	
	$= \frac{1}{4}[(0 + \pi) - (0 + 0)]$		
	$= \frac{\pi}{4}$	A1	
		(5)	
		[16]	

Qu.	Solution	Mark	Notes
7.	$\lvert z\rvert = \sqrt{11^2 + (-2)^2} = 5\sqrt{5}$	B1	si
	$\arg(z) = \tan^{-1}\left(-\frac{2}{11}\right) = -0.179(85\ldots)$	B1	si
	Therefore cube roots:		
	1st root: $\sqrt[3]{5\sqrt{5}}(\cos -0.05995 + \mathrm{i}\sin -0.05995)$	M1	Use of de Moivre. FT $\lvert z\rvert$ and $\arg(z)$
	$= 2.232 - 0.134\mathrm{i}$	A1	
	2nd root: $\sqrt[3]{5\sqrt{5}}(\cos 2.0344 + \mathrm{i}\sin 2.0344)$	M1	$+\frac{2\pi}{3}$
	$= -1.000 + 2.000\mathrm{i}$	A1	
	3rd root: $\sqrt[3]{5\sqrt{5}}(\cos 4.1288 + \mathrm{i}\sin 4.1288)$		
	$= -1.232 - 1.866\mathrm{i}$	A1	
		[7]	Note: $\sqrt[3]{5\sqrt{5}} = \sqrt{5}$
8. a)	Mean value = $\frac{1}{2-0}\int_0^2 \left(\frac{1}{\sqrt{x^2+4x+3}}\right) \mathrm{d}x$	M1	Condone omission of 1/(b-a)
	$\frac{1}{2}\int_0^2 \left(\frac{1}{\sqrt{(x+2)^2-1}}\right) \mathrm{d}x$	m1 A1	Completing the square Must include 1/(b-a)
	$= \frac{1}{2}\left[\cosh^{-1}\left(\frac{x+2}{1}\right)\right]_0^2$	A1	or $\frac{1}{2}\ln\left(x + 2 + \sqrt{((x+2)^2 - 1)}\right)$ or $\frac{1}{2}\ln\left(x + 2 + \sqrt{x^2 + 4x + 3}\right)$ Correct integration of $\frac{1}{\sqrt{x^2 \pm a^2}}$
	$= 0.373$	A1	cao
		(5)	No marks awarded for answer only.
b)	METHOD 1:		
	Volume = $\pi\int_0^2 \left(\frac{1}{\sqrt{x^2+4x+3}}\right)^2 \mathrm{d}x$	M1	Condone omission of π
	$= \pi\int_0^2 \frac{1}{x^2+4x+3}\mathrm{d}x$		
	$= \pi\int_0^2 \frac{1}{(x+2)^2 - 1}\mathrm{d}x$	A1	Fully correct Must include π and limits
	$= \frac{\pi}{2}\left[\ln\left\lvert\frac{x+2-1}{x+2+1}\right\rvert\right]_0^2$	m1 A1	Limits not required Correct integration
	$= \frac{\pi}{2}\left[\ln\left\lvert\frac{x+1}{x+3}\right\rvert\right]_0^2$		
	$= \frac{\pi}{2}\left[\ln\frac{3}{5} - \ln\frac{1}{3}\right]$	m1	Correct use of limits
	$= \frac{\pi}{2}\ln\frac{9}{5}$	A1	oe Must be exact Mark final answer

Qu.	Solution	Mark	Notes
8. b)	METHOD 2: $\left(\frac{1}{\sqrt{x^2+4x+3}}\right)^2 = \frac{1}{x^2+4x+3}$		
	$\frac{1}{x^2+4x+3} = \frac{A}{x+1} + \frac{B}{x+3}$	(M1)	Use of partial fractions
	$1 = A(x+3) + B(x+1)$		
	When $x = -1,\ 1 = 2A$ $\therefore A = \frac{1}{2}$		
	When $x = -3, 1 = -2B$ $\therefore B = -\frac{1}{2}$	(A1)	Both values
	Volume = $\pi \int_0^2 \left(\frac{1}{\sqrt{x^2+4x+3}}\right)^2 \mathrm{d}x$	(M1)	Condone omission of π
	$= \pi \int_0^2 \left(\frac{1}{2(x+1)} - \frac{1}{2(x+3)}\right) \mathrm{d}x$ $= \pi \left[\frac{1}{2}\ln\lvert x+1\rvert - \frac{1}{2}\ln\lvert x+3\rvert\right]_0^2$	(A1)	Correct integration. Must include π
	$= \pi\left[\left(\frac{1}{2}\ln 3 - \frac{1}{2}\ln 5\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{2}\ln 3\right)\right]$	(m1)	Use of limits
	$= \frac{\pi}{2}\ln\frac{9}{5}$	(A1)	oe Must be exact Mark final answer
		(6)	
		[11]	

Qu.	Solution	Mark	Notes
9.	Dividing both sides by $(x+1)$: $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{5y}{x+1} = x + 1$	M1	
	Integrating factor: $\mathrm{e}^{\int \frac{5}{x+1}\mathrm{d}x}$ $= \mathrm{e}^{5\ln(x+1)} = (x+1)^5$	m1 A1	 cao
	Multiplying both sides: $(x+1)^5 \frac{\mathrm{d}y}{\mathrm{d}x} + 5y(x+1)^4 = (x+1)^6$	M1	
	Integrating: $(x+1)^5 y = \frac{(x+1)^7}{7} + c$	m1 A1	 Or equivalent, cao
	Substituting $32 \times \frac{1}{4} = \frac{128}{7} + c \rightarrow c = \frac{-72}{7}$	B1	
	Solution $(x+1)^5 y = \frac{(x+1)^7}{7} - \frac{72}{7}$		
	When $x = 0$, $y = \frac{1}{7} - \frac{72}{7}$ $\rightarrow y = -\frac{71}{7} (-10.\dot{1}4285\dot{7})$	B1	
		[8]	

Qu.	Solution	Mark	Notes
10.a)	METHOD 1: Writing $y = \sin^{-1}(2x+5)$ $\therefore \sin y = 2x+5$ $\frac{1}{2}(\sin y - 5) = x$		
	Differentiating, $\frac{dx}{dy} = \frac{1}{2}\cos y$	M1 A1	Rearrange and attempt to differentiate
	Since, $\sin^2 y + \cos^2 y = 1$ $(2x+5)^2 + \cos^2 y = 1$ $\cos^2 y = 1-(2x+5)^2$	B1	si
	Therefore, $\frac{dx}{dy} = (\pm)\frac{1}{2}\sqrt{1-(2x+5)^2}$		
	$\frac{dy}{dx} = (\pm)\frac{2}{\sqrt{1-(2x+5)^2}}$	A1	
	$\frac{dy}{dx} = \frac{2}{\sqrt{1-(2x+5)^2}}$ AND valid justification, eg. reference to the gradient of the graph.	A1	
	METHOD 2: Writing $y = \sin^{-1}(2x+5)$ $\therefore \sin y = 2x+5$		
	Differentiating, $\cos y \frac{dy}{dx} = 2$	(M1)	Attempt to differentiate
	$\frac{dy}{dx} = \frac{2}{\cos y}$	(A1)	
	Since, $\sin^2 y + \cos^2 y = 1$ $(2x+5)^2 + \cos^2 y = 1$ $\cos^2 y = 1-(2x+5)^2$	(B1)	si
	Therefore, $\frac{dy}{dx} = (\pm)\frac{2}{\sqrt{1-(2x+5)^2}}$	(A1)	
	$\frac{dy}{dx} = \frac{2}{\sqrt{1-(2x+5)^2}}$ AND valid justification, eg. reference to the gradient of the graph.	(A1)	
		(5)	

Qu.	Solution	Mark	Notes
10.b)	$1-(2x+5)^2=-4x^2-20x-24$		
	METHOD 1:		
	When $-4x^2-20x-24=0$	M1	
	$x=-2$ or $x=-3$	A1	
	Therefore the range of valid values is $-3<x<-2$	A1	
	METHOD 2:		
	Valid when $1-(2x+5)^2>0$	(M1)	
	$-4x^2-20x-24>0$		
	$4x^2+20x+24<0$		
	$4(x+3)(x+2)<0$	(A1)	
	Therefore, $-3<x<-2$	(A1)	
	METHOD 3:		
	$\sin y=2x+5$		
	Valid when $-1\leq 2x+5\leq 1$	(M1)	
	$-3\leq x\leq -2$	(A1)	
	Therefore (due to fraction), $-3<x<-2$	(A1)	
		(3)	
		[8]	

Qu.	Solution	Mark	Notes
11.a)	METHOD 1: Using integration by parts: $I = \left[\frac{1}{2}\mathrm{e}^{2x}\sinh x\right]_{-2}^{0} - \int_{-2}^{0}\frac{1}{2}\mathrm{e}^{2x}\cosh x\,\mathrm{d}x$	M1 A1	Attempt by parts
	$I = \left[\frac{1}{2}\mathrm{e}^{2x}\sinh x - \frac{1}{4}\mathrm{e}^{2x}\cosh x\right]_{-2}^{0} + \int_{-2}^{0}\frac{1}{4}\mathrm{e}^{2x}\sinh x\,\mathrm{d}x$	A1	
	$\frac{3}{4}I = \left[\frac{1}{2}\mathrm{e}^{2x}\sinh x - \frac{1}{4}\mathrm{e}^{2x}\cosh x\right]_{-2}^{0}$	A1	Fully correct including limits
	$I = \frac{-0.199599\ldots}{\frac{3}{4}} = -0.266\ldots$	A1	
	METHOD 2: Using integration by parts: $I = \left[\mathrm{e}^{2x}\cosh x\right]_{-2}^{0} - \int_{-2}^{0}2\mathrm{e}^{2x}\cosh x\,\mathrm{d}x$	(M1) (A1)	Attempt by parts
	$I = \left[\mathrm{e}^{2x}\cosh x - 2\mathrm{e}^{2x}\sinh x\right]_{-2}^{0} + \int_{-2}^{0}4\mathrm{e}^{2x}\sinh x\,\mathrm{d}x$	(A1)	
	$-3I = \left[\mathrm{e}^{2x}\cosh x - 2\mathrm{e}^{2x}\sinh x\right]_{-2}^{0}$	(A1)	Fully correct including limits
	$I = \frac{0.798236\ldots}{-3} = -0.266\ldots$	(A1)	
	METHOD 3: Using substitution: $I = \int_{-2}^{0}\mathrm{e}^{2x}\left(\frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2}\right)\mathrm{d}x$	(M1)	Substitution of $\frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2}$
	$I = \frac{1}{2}\int_{-2}^{0}(\mathrm{e}^{3x} - \mathrm{e}^{x})\mathrm{d}x$	(A1)	
	$I = \frac{1}{2}\left[\frac{1}{3}\mathrm{e}^{3x} - \mathrm{e}^{x}\right]_{-2}^{0}$	(A1)	Both terms
	$I = \frac{1}{2}\left[\left(\frac{1}{3} - 1\right) - \left(\frac{1}{3}\mathrm{e}^{-6} - \mathrm{e}^{-2}\right)\right]$	(A1)	Correct use of limits
	$= -0.266\ldots$	(A1)	
		(5)	No marks awarded for answers only.

Qu.	Solution	Mark	Notes
11.b)	$\frac{5}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$	M1	
	$5 = A(x^2+9) + (Bx+C)(x-1)$	A1	
	When $x = 1, 5 = 10A$ $\therefore A = \frac{1}{2}$		
	When $x = 0, 5 = 9A - C$ $\therefore C = -\frac{1}{2}$		
	Co-efficients of x^2: $0 = A + B$ $\therefore B = -\frac{1}{2}$	A2	A1 any 2 coefficients
	$\int_{1.5}^{3} \frac{5}{(x-1)(x^2+9)}\,\mathrm{d}x$		
	$\int_{1.5}^{3} \left(\frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}x - \frac{1}{2}}{x^2+9}\right)\mathrm{d}x$	M1	FT their A, B, C provided at least two non-zero values
	$= \int_{1.5}^{3} \left(\frac{1}{2(x-1)} - \frac{x}{2(x^2+9)} - \frac{1}{2(x^2+9)}\right)\mathrm{d}x$	A1	Three terms
	$= \left[\frac{1}{2}\ln\lvert x-1\rvert - \frac{1}{4}\ln\lvert x^2+9\rvert - \frac{1}{6}\tan^{-1}\frac{x}{3}\right]_{1.5}^{3}$	A2	Three terms all correct (A1 any 1 term)
	$= 0.522\ (0211808\ldots)$	A1	cao
		(9) **[14]**	No marks awarded for answers only.

Qu.	Solution	Mark	Notes
12.	$2\cos\dfrac{4\theta+2\theta}{2}\cos\dfrac{4\theta-2\theta}{2}=\cos\theta$	M1	M0 no working
	$2\cos 3\theta\cos\theta-\cos\theta=0$	A1	
	$\cos\theta\,(2\cos 3\theta-1)=0$		
	$\therefore\cos\theta=0,\qquad \cos 3\theta=\frac{1}{2}$	A1	For both
	$\theta=\frac{\pi}{2}+n\pi$	m1 A1	Use of $\cos$ general solution oe
	$3\theta=2n\pi\pm\frac{\pi}{3}$	(m1)	If not awarded previously
	$\theta=\frac{2n\pi}{3}\pm\frac{\pi}{9}$	A1	oe
		[6]	
13.a) i)	METHOD 1: Rearrange first equation: $y=\dfrac{1}{10}\left(\dfrac{dx}{dt}-3x\right)$	M1	
	Substituting into second equation: $\dfrac{d}{dt}\left(\dfrac{1}{10}\left(\dfrac{dx}{dt}-3x\right)\right)=x+6\left(\dfrac{1}{10}\left(\dfrac{dx}{dt}-3x\right)\right)$	M1	
	$\dfrac{1}{10}\dfrac{d^2x}{dt^2}-\dfrac{3}{10}\dfrac{dx}{dt}=x+\dfrac{6}{10}\dfrac{dx}{dt}-\dfrac{18}{10}x$	A1	
	$\dfrac{1}{10}\dfrac{d^2x}{dt^2}-\dfrac{9}{10}\dfrac{dx}{dt}+\dfrac{8}{10}x=0$		
	$\dfrac{d^2x}{dt^2}-9\dfrac{dx}{dt}+8x=0$	A1	Convincing
	METHOD 2: Differentiating: $\frac{d^2x}{dt^2}=3\frac{dx}{dt}+10\frac{dy}{dt}$	(M1)	
	$\frac{d^2x}{dt^2}=3\frac{dx}{dt}+10(x+6y)$		or $\frac{d^2x}{dt^2}=3\frac{dx}{dt}+10x+60y$
	Substituting for y $\frac{d^2x}{dt^2}=3\frac{dx}{dt}+10x+6\left(\frac{dx}{dt}-3x\right)$	(M1)	
	$\frac{d^2x}{dt^2}=3\frac{dx}{dt}+10x+6\frac{dx}{dt}-18x$	(A1)	
	$\frac{d^2x}{dt^2}-9\frac{dx}{dt}+8x=0$	(A1)	Convincing
		(4)	

Qu.	Solution	Mark	Notes
13.a) ii)	The auxiliary equation is: $m^2 - 9m + 8 = 0$ $(m-1)(m-8) = 0$ $m = 1$ or $m = 8$ Therefore, the general solution is: $x = A\mathrm{e}^{t} + B\mathrm{e}^{8t}$	B1 B1 B1 **(3)**	
b)	Differentiating gives: $\frac{\mathrm{d}x}{\mathrm{d}t} = A\mathrm{e}^{t} + 8B\mathrm{e}^{8t}$ Therefore, $y = \frac{1}{10}\left((A\mathrm{e}^{t} + 8B\mathrm{e}^{8t}) - 3(A\mathrm{e}^{t} + B\mathrm{e}^{8t})\right)$ $y = \frac{1}{10}(-2A\mathrm{e}^{t} + 5B\mathrm{e}^{8t})$	M1 A1 m1 A1 **(4)**	FT their x provided 2 real roots cao
c)	When $t = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 5$ $\therefore 5 = A + 8B$ Also, $y = 4x$ $\frac{1}{10}(-2A\mathrm{e}^{t} + 5B\mathrm{e}^{8t}) = 4(A\mathrm{e}^{t} + B\mathrm{e}^{8t})$ When $t = 0$, $\frac{1}{10}(-2A + 5B) = 4(A + B)$ $\therefore A = -\frac{35}{42}B \quad \left(= -\frac{5}{6}B\right)$ Also, $5 = -\frac{5}{6}B + 8B$ $B = \frac{30}{43}$ $A = -\frac{25}{43}$ Therefore, the particular solution is: $x = -\frac{25}{43}\mathrm{e}^{t} + \frac{30}{43}\mathrm{e}^{8t}$	B1 M1 A1 M1 A1 A1 **(6)** **[17]**	 Equating y and $4x$ Correct equation Attempt to remove one variable Both values FT A, B above