A2 Further Mathematics Unit 4: Further Pure Mathematics B
General instructions for marking GCE Mathematics

1. The mark scheme should be applied precisely and no departure made from it. Marks should be awarded directly as indicated and no further subdivision made.

2. Marking Abbreviations
The following may be used in marking schemes or in the marking of scripts to indicate reasons for the marks awarded.

cao = correct answer only
MR = misread
PA = premature approximation
bod = benefit of doubt
oe = or equivalent
si = seen or implied
ISW = ignore subsequent working

F.T. = follow through (✓ indicates correct working following an error and ✗ indicates a further error has been made)

Anything given in brackets in the marking scheme is expected but, not required, to gain credit.

3. Premature Approximation
A candidate who approximates prematurely and then proceeds correctly to a final answer loses 1 mark as directed by the Principal Examiner.

4. Misreads
When the data of a question is misread in such a way as not to alter the aim or difficulty of a question, follow through the working and allot marks for the candidates' answers as on the scheme using the new data.
This is only applicable if a wrong value, is used consistently throughout a solution; if the correct value appears anywhere, the solution is not classed as MR (but may, of course, still earn other marks).

5. Marking codes

- 'M' marks are awarded for any correct method applied to appropriate working, even though a numerical error may be involved. Once earned they cannot be lost.
- 'm' marks are dependant method marks. They are only given if the relevant previous 'M' mark has been earned.
- 'A' marks are given for a numerically correct stage, for a correct result or for an answer lying within a specified range. They are only given if the relevant M/m mark has been earned either explicitly or by inference from the correct answer.
- 'B' marks are independent of method and are usually awarded for an accurate result or statement.
- 'S' marks are awarded for strategy
- 'E' marks are awarded for explanation
- 'U' marks are awarded for units
- 'P' marks are awarded for plotting points
- 'C' marks are awarded for drawing curves

A2 Further Mathematics Unit 4: Further Pure Mathematics B

Solutions and Mark Scheme

Qu. No.	Solution	Mark	AO	Notes
1.(a)	$\int_0^{\infty} \frac{\mathrm{d}x}{(1+x)^5} = -\frac{1}{4}\left[\frac{1}{(1+x)^4}\right]_0^{\infty}$	M1	AO1	
	$= -\frac{1}{4}(0-1)$	A1	AO1	
	$=\frac{1}{4}$	A1	AO1	
(b)	$\mathrm{d}u = \frac{\mathrm{d}x}{x}; [2,\infty) \to [\ln 2, \infty)$	B1	AO1	
	Integral = $\int_{\ln 2}^{\infty} \frac{\mathrm{d}u}{u}$	M1	AO1	
	$= [\ln u]_{\ln 2}^{\infty}$	A1	AO1	
	$\to \infty$ because $\ln u \to \infty$	A1	AO1	
		[7]		
2.	Attempting to complete the square	M1	AO3	Award M0 for unsupported working
	Integral $= \int_0^1 \frac{\mathrm{d}x}{\sqrt{2(x+1)^2+4}}$	A1	AO3	
	$= \frac{1}{\sqrt{2}}\int_0^1 \frac{\mathrm{d}x}{\sqrt{(x+1)^2+2}}$	A1	AO3	
	$= \frac{1}{\sqrt{2}}\left[\sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right]_0^1$	A1	AO3	
	$= \frac{1}{\sqrt{2}}\left(\sinh^{-1}\left(\frac{2}{\sqrt{2}}\right) - \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$	A1	AO3	
	$= 0.345 \ (0.344882\ldots)$	A1	AO3	
		[6]		

Qu. No.	Solution	Mark	AO	Notes
3.	Area = $\frac{1}{2}\int r^2 \mathrm{d}\theta$	M1	AO1	
	$= \frac{9}{2}\int_0^{\pi}(4+4\cos\theta+\cos^2\theta)\mathrm{d}\theta$	A1	AO1	
	$= \frac{9}{2}\int_0^{\pi}\left(\frac{9}{2}+4\cos\theta+\frac{\cos 2\theta}{2}\right)$	A1	AO1	
	$= \frac{9}{2}\left[\frac{9}{2}\theta+4\sin\theta+\frac{\sin 2\theta}{4}\right]_0^{\pi}$	A1	AO1	
	$= \frac{81\pi}{4}$	A1 **[5]**	AO1	
4.	$\lvert z\rvert = \sqrt{13}$	B1	AO3	
	$\arg(z) = \tan^{-1}1.5 = 0.98279...$	B1	AO3	
	$z = \sqrt{13}(\cos 0.98279.. + \mathrm{i}\sin 0.98279..)$	M1	AO3	
	First cube root $= 13^{1/6}(\cos 0.32759... + \mathrm{i}\sin 0.32759...)$	m1	AO3	
	$= 1.45 + 0.493\mathrm{i}$	A1	AO3	
	Second cube root $= 13^{1/6}(\cos(0.32759... + 2\pi/3) + \mathrm{i}\sin(0.32759... + 2\pi/3))$	M1	AO3	
	$= -1.15 + 1.01\mathrm{i}$	A1	AO3	
	Third cube root $= 13^{1/6}(\cos(0.32759... + 4\pi/3) + \mathrm{i}\sin(0.32759... + 4\pi/3))$	M1	AO3	
	$= -0.298 - 1.50\mathrm{i}$	A1 **[9]**	AO3	

Qu. No.	Solution	Mark	AO	Notes
5.	Rewrite the equation in the form $\cos 3\theta + 2\cos 2\theta \cos 3\theta = 0$ $\cos 3\theta(1 + 2\cos 2\theta) = 0$	 M1 A1	 AO1 AO1	
	Either $\cos 3\theta = 0$	M1	AO1	
	$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$	A1	AO1	
	$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$	A1	AO1	
	Or $\cos 2\theta = -\frac{1}{2}$	M1	AO1	
	$2\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$	A1	AO1	
	$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$	A1	AO1	
		[8]		
6.(a)(i)	$\mathrm{adj}(\mathbf{M}) = \begin{bmatrix} 11 & 1 & -7 \\ 1 & 1 & -1 \\ -7 & -1 & 5 \end{bmatrix}$	M1 A1	AO1 AO1	Award M1 if at least 5 correct
(ii)	$\det(\mathbf{M}) = 2\times(15–4) + 1\times(6–5) + 3\times(2–9)$ $= 2$	M1 A1	AO1 AO1	
	$\mathbf{M}^{-1} = \frac{1}{2}\begin{bmatrix} 11 & 1 & -7 \\ 1 & 1 & -1 \\ -7 & -1 & 5 \end{bmatrix}$	B1	AO1	
(b)	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 11 & 1 & -7 \\ 1 & 1 & -1 \\ -7 & -1 & 5 \end{bmatrix}\begin{bmatrix} 13 \\ 13 \\ 22 \end{bmatrix}$	M1	AO1	
	$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$	A1	AO1	
		[7]		

Qu. No.	Solution	Mark	AO	Notes
7.(a)	Let $\dfrac{8x^2+x+5}{(2x+1)(x^2+3)} = \dfrac{A}{2x+1} + \dfrac{Bx+C}{x^2+3}$	M1	AO1	
	$= \dfrac{A(x^2+3)+(Bx+C)(2x+1)}{(2x+1)(x^2+3)}$ $A = 2, B = 3, C = -1$	A1 A1 A1	AO1 AO1 AO1	A1 each constant
(b)	Integral $= \left(\int_2^3 \dfrac{2}{2x+1} + \dfrac{3x}{x^2+3} - \dfrac{1}{x^2+3} \right) \mathrm{d}x$	M1	AO1	Award M0 for work unsupported
	$= \left[\ln(2x+1) + \dfrac{3}{2}\ln(x^2+3) - \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right) \right]_2^3$	A1 A1 A1	AO1 AO1 AO1	A1 each integral
	$=$ $\ln 7 + \dfrac{3}{2}\ln 12 - \dfrac{1}{\sqrt{3}}\tan^{-1}\sqrt{3} - \ln 5 - \dfrac{3}{2}\ln 7 + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$	A1	AO1	
	$= 1.035$	A1	AO1	
		[10]		
8.(a)	Capacity $= \pi\int_1^9 x^2 \mathrm{d}y$	M1	AO3	
	$= \pi\int_1^9 (y-1)^{2/3}\mathrm{d}y$	A1	AO3	
	$= \pi\left[\dfrac{3}{5}(y-1)^{5/3}\right]_1^9$	A1	AO3	
	$= \dfrac{3\pi}{5}(32-0)$	A1	AO3	
	$= 60.3(1857….)$	A1	AO3	
(b)	Capacity $= \pi\int_1^a (y-1)^{2/3}\mathrm{d}y$	M1	AO3	
	$= \pi\left[\dfrac{3}{5}(y-1)^{5/3}\right]_1^a$	A1	AO3	
	$= \dfrac{3\pi}{5}(a-1)^{5/3}$	A1	AO3	
	Attempting to solve $\dfrac{3\pi}{5}(a-1)^{5/3} = 25$	M1	AO3	
	$a = 5.72 \ (5.71610...)$	A1	AO3	
		[10]		

Qu. No.	Solution	Mark	AO	Notes
9.(a)	Putting $n = 1$, the proposition gives $\cos\theta + \mathrm{i}\sin\theta = \cos\theta + \mathrm{i}\sin\theta$ which is true	B1	AO2	
	Let the proposition be true for $n = k$, ie $[\cos\theta + \mathrm{i}\sin\theta]^k = \cos k\theta + \mathrm{i}\sin k\theta$	M1	AO2	
	Consider (for $n = k + 1$) $(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = (\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + i\sin\theta)$	M1	AO2	
	$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$	A1	AO2	
	$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta\cos\theta + \sin\theta\cos k\theta)$	A1	AO2	
	$= \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta$	A1	AO2	
	Therefore true for $n = k \Rightarrow$ true for $n = k + 1$ and since true for n =1 the proposition is proved by induction.	A1	AO2	
(b)(i)	Consider $\cos 5\theta + \mathrm{i}\sin 5\theta = (\cos\theta + \mathrm{i}\sin\theta)^5$	M1	AO2	
	$= \mathrm{i}(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$ + real terms	A1	AO2	
	It follows equating imaginary terms that $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$	A1	AO2	
	$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$	A1	AO2	
	$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$	A1	AO2	
(ii)	$\dfrac{\sin 5\theta}{\sin\theta} = 16\sin^4\theta - 20\sin^2\theta + 5$	M1	AO1	
	$\to 5$ as $\theta \to 0$	A1	AO1	
		[14]		

Qu. No.	Solution	Mark	AO	Notes
10.(a)	Integrating factor $= \mathrm{e}^{\int 2\tan x\mathrm{d}x}$	M1	AO1	
	$= \mathrm{e}^{2\ln\sec x}$	A1	AO1	
	$= \mathrm{e}^{\ln\sec^2 x}$	A1	AO1	
	$= \sec^2 x$	A1	AO1	
(b)	Applying the integrating factor,			
	$\sec^2 x\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y\tan x\sec^2 x = \sin x\sec^2 x$	M1	AO1	
	$= \dfrac{\sin x}{\cos^2 x}$ (or $\sec x\tan x$)	A1	AO1	
	Integrating,			
	$y\sec^2 x = \sec x + C$	A1 A1	AO1 AO1	A1 each side
	$0 = \sqrt{2} + C$	M1	AO1	
	$C = -\sqrt{2}$	A1	AO1	
	The solution is $y = \cos x - \sqrt{2}\cos^2 x$	A1	AO1	
		[11]		

Qu. No.	Solution	Mark	AO	Notes
11.(a)	Let $y = \tanh^{-1} x$ so $x = \tanh y$			
	$= \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$	M1	AO2	
	$xe^y + xe^{-y} = e^y - e^{-y}$	A1	AO2	
	$e^{2y} = \dfrac{1+x}{1-x}$	A1	AO2	
	$y = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$	A1	AO2	
(b)	$a\cosh x + b\sinh x \equiv r\cosh(x+\alpha)$	M1	AO2	
	$= r\cosh x\cosh\alpha + r\sinh x\sinh\alpha$	A1	AO2	
	Equating like terms,			
	$r\cosh\alpha = a$			
	$r\sinh\alpha = b$	A1	AO2	
	Dividing,			
	$\tanh\alpha = \dfrac{b}{a}$	M1	AO2	
	$\alpha = \tanh^{-1}\left(\dfrac{b}{a}\right)$	A1	AO2	
	$= \dfrac{1}{2}\ln\left(\dfrac{1+b/a}{1-b/a}\right) = \dfrac{1}{2}\ln\left(\dfrac{a+b}{a-b}\right)$			
	Squaring and subtracting the above equations,			
	$r^2(\cosh^2\alpha - \sinh^2\alpha) = a^2 - b^2$	M1	AO1	
	$r = \sqrt{a^2 - b^2}$	A1	AO1	
(c)	Here $r = 3$	B1	AO1	
	$\alpha = \dfrac{1}{2}\ln 9 = \ln 3$	B1	AO1	
	The equation simplifies to $3\cosh(x + \ln 3) = 10$	B1	AO1	
	$x + \ln 3 = (\pm)\cosh^{-1}\left(\dfrac{10}{3}\right)$	M1	AO1	
	$x = 0.775$	A1	AO1	
	or $x = -2.97$	A1	AO1	
		[17]		

Qu. No.	Solution	Mark	AO	Notes
12.(a)	$f'(x) = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x$	B1	AO2	
	$f''(x) = \mathrm{e}^x \cos x - \mathrm{e}^x \sin x - \mathrm{e}^x \sin x - \mathrm{e}^x \cos x$	B1	AO2	
	$= -2\mathrm{e}^x \sin x$			
(b)	$f'''(x) = -2\mathrm{e}^x \sin x - 2\mathrm{e}^x \cos x$	B1	AO1	
	$f^{(4)}(x) = -2\mathrm{e}^x \sin x - 2\mathrm{e}^x \cos x - 2\mathrm{e}^x \cos x + 2\mathrm{e}^x \sin x$	B1	AO1	
	$(= -4\mathrm{e}^x \cos x)$			
	$f(0) = 1, f'(0) = 1, f''(0) = 0$	B1	AO1	
	$f'''(0) = -2, f^{(4)}(0) = -4$	B1	AO1	
	The Maclaurin series is			
	$\mathrm{e}^x \cos x = 1 + x - \frac{2x^3}{6} - \frac{4x^4}{24} + \ldots$	M1	AO1	
	$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \ldots$	A1	AO1	
(c)	Valid attempt at differentiating both sides,	M1	AO1	
	$\mathrm{e}^x \cos x - \mathrm{e}^x \sin x = 1 - x^2 - \frac{2x^3}{3} + \ldots$	A1	AO1	
	$\mathrm{e}^x \sin x = 1 + x - \frac{x^3}{3} - 1 + x^2 + \frac{2x^3}{3} + \ldots$	A1	AO1	
	$= x + x^2 + \frac{x^3}{3} + \ldots$	A1	AO1	
(d)	Replacing $\mathrm{e}^x \sin x$ by its series,			
	$10\left(x + x^2 + \frac{x^3}{3}\right) - 11x = 0$	M1	AO3	
	$10x^3 + 30x^2 - 3x = 0$	A1	AO3	
	$x = \frac{-30 + \sqrt{900 + 120}}{20}$	m1	AO3	
	$= 0.097$	A1	AO3	
		[16]		