Pearson Edexcel
Level 3 Advanced Subsidiary
GCE in Further Mathematics (8FM0)

Pearson Edexcel
Level 3 Advanced
GCE in Further Mathematics (9FM0)

June 2019 – Core Pure Exemplar

Student answers with examiner comments

First teaching from September 2017

Contents

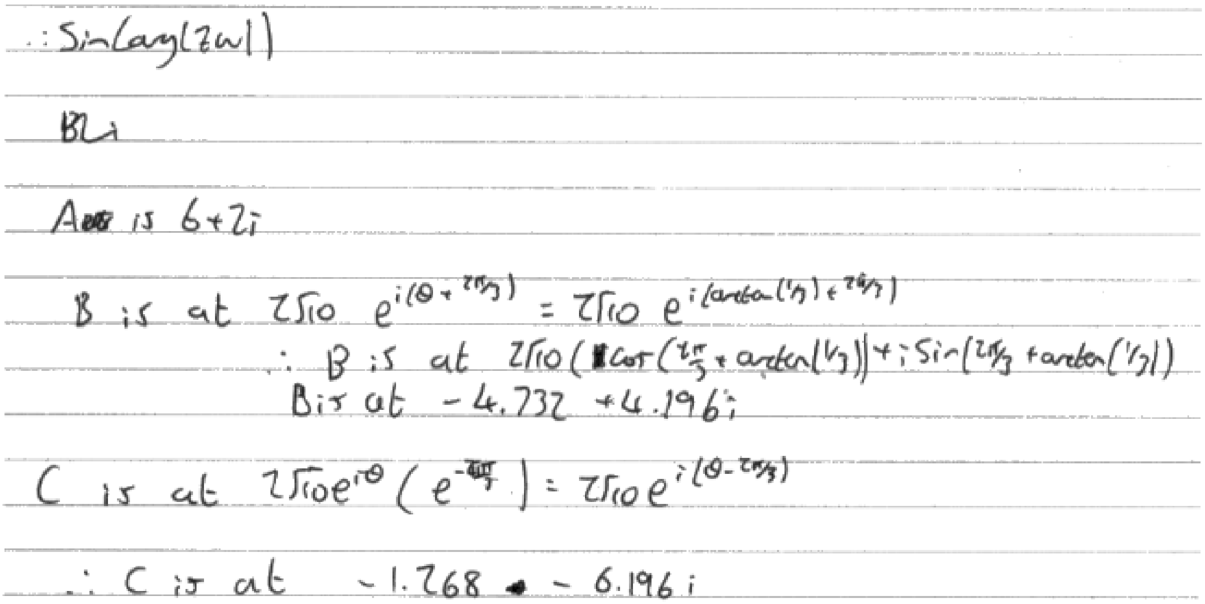
About this booklet

This booklet has been produced to support mathematics teachers delivering the new Pearson Edexcel Level 3 Advanced Subsidiary and Advance Level GCE in Further Mathematics specification (8FM0 & 9FM0). The booklet looks at questions from the AS and A Level Further Mathematics - Core Pure June 2019 Examination Papers. It shows student responses to questions, and how the examining team follow the mark schemes to demonstrate how the students would be awarded marks on these questions.

How to use this booklet

Our examining team have selected student responses to all questions from the June 2019 Examination Papers. Following each question, you will find the mark scheme for that question and then a range of student responses with accompanying examiner comments on how the mark scheme has been applied and the marks awarded, and on common errors for this sort of question.

Student Response



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Examiner Comments

In part (a), this candidate adopts a correct strategy for finding the points B and C by multiplying the exponential form of 6 + 2i by $e^{\frac{2\pi}{3}}$ and $e^{-\frac{2\pi}{3}}$ but does not obtain any of the required values in the required exact form.

There is no attempt at part (b).

Examiner commentary on the student response

Marks awarded for the question

AS Further Mathematics – Core Pure (8FM0 01)

Exemplar Question 1

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1.

$$\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$$

(*a*) Show that the matrix **M** is non-singular.

(2)

The transformation T of the plane is represented by the matrix **M**.

The triangle R is transformed to the triangle S by the transformation T.

Given that the area of S is 63 square units,

(*b*) find the area of R.

(2)

(*c*) Show that the line $y = 2x$ is invariant under the transformation T.

(2)

(Total for Question 1 is 6 marks)

Mean Score 4.1 out of 6

Examiner comment

This question is assessing the student's understanding of transformation matrices (3.3,3.4).

Part (a) was the most successful part of the question. Most candidates understood that they needed to find the determinant and compare to zero. There was some poor arithmetic with negative numbers seen and some failed to reach a conclusion, simply stating det non zero. The majority of responses were fully correct for this part.

For part (b), again most responses were correct here, with the follow through seldom needed. Some students did not seem to know the connection between determinants and area scale factors, while others neglected to use the modulus of the determinant, giving a negative answer. A few cases of confusing the areas of *R* and *S* were also seen from students, though these were infrequent.

Part (c) caused the most difficulty in this question. The main error was that students had not understood the difference between invariant points and invariant lines, attempting to solve **Ax** = **x**. Other students tried to find the equation of the invariant line from first principles (it might be noted this method was required on a specimen paper) rather than simply checking that y=2x was invariant. Such attempt usually met with success, but the method is more cumbersome that was required.

Mark Scheme

Question	Scheme	Marks	AOs
1 (a)	(det(**M**) =) $(4)(-7)-(2)(-5)$	M1	1.1a
	M is non-singular because $\det(\mathbf{M})=-18$ and so $\det(\mathbf{M})\neq 0$	A1	2.4
		(2)	
(b)	Area $R=\dfrac{\text{Area } S}{(\pm)\lvert\det \mathbf{M}\rvert}=\ldots$	M1	1.2
	$\text{Area}(R)=\dfrac{63}{\lvert -18\rvert}=\dfrac{7}{2}$ oe	A1ft	1.1b
		(2)	
(c)	$\begin{pmatrix}4 & -5\\ 2 & -7\end{pmatrix}\begin{pmatrix}x\\ 2x\end{pmatrix}=\begin{pmatrix}4x-10x\\ 2x-14x\end{pmatrix}$	M1	1.1b
	$=\begin{pmatrix}-6x\\ -12x\end{pmatrix}$ and so all points on $y=2x$ map to points on $y=2x$, hence the line is invariant. OR $=-6\begin{pmatrix}x\\ 2x\end{pmatrix}$ hence $y=2x$ is invariant.	A1	2.1
		(2)	
	(6 marks)		

Notes

(a)	M1	An attempt to find $\det(\mathbf{M})$. Just the calculation is sufficient. Site of -18 implies this mark, which may be embedded in an attempt at the inverse..
	A1	$\det(\mathbf{M})=-18$ **and** reference to zero, e.g. $-18\neq 0$ **and** conclusion. The conclusion may precede finding the determinant (e.g. "Non-singular if det(**M**) ≠ 0, det(**M**) = −18 ≠ 0" is sufficient or accept "Non-singular if det(**M**)≠ 0, det(**M**) = −18, therefore non-singular" or some other indication of conclusion.) Need not mention "det(**M**)" to gain both marks here, a correct calculation, statement −18≠0, and conclusion hence **M** is non-singular can gain M1A1.
(b)	M1	Recalls determinant is needed for area scale factor by dividing 63 by ±their determinant.
	A1ft	$\dfrac{7}{2}$ or follow through $\dfrac{63}{\lvert\text{their det}\rvert}$. Must be positive and should be simplified to single fraction or exact decimal. (Allow if made positive following division by a negative determinant.)
(c)	M1	Attempts the matrix multiplication shown or with equivalent, e.g $\begin{pmatrix}\frac{1}{2}y\\ y\end{pmatrix}$. May use $\begin{pmatrix}x\\ y\end{pmatrix}$ **and** substitute $y=2x$ later and this is fine for the method.
	A1	Correct multiplication and working leading to conclusion that the line is invariant. If the −6 is not extracted, they must make reference to image points being on line $y=2x$. If the −6 is extracted to show it is a multiple of $\begin{pmatrix}x\\ 2x\end{pmatrix}$ followed by a conclusion "invariant" as minimum.

Notes Continued				
Alt for (c)	$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{-18}\begin{pmatrix} -7 & 5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \frac{-1}{18}\begin{pmatrix} -7x+10x \\ -2x+8x \end{pmatrix}$		**M1**	1.1b
	$= \frac{-1}{18}\begin{pmatrix} 3x \\ 6x \end{pmatrix}\left(= \frac{-1}{6}\begin{pmatrix} x \\ 2x \end{pmatrix}\right) \Rightarrow b = 2a$ so points on line y = 2x map to points on y= 2x, hence it is invariant.		**A1**	2.1
	Marks as per main scheme,			
Alt 2	(Since linear transformations map straight lines to straight lines…) E.g. $(1, 2)$ is on line y = 2x, and $\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4-10 \\ 2-14 \end{pmatrix}$		**M1**	1.1b
	$= \begin{pmatrix} -6 \\ -12 \end{pmatrix}$, which is also on the line y=2x, hence as (0,0) and (1,2) both map to points on y = 2x (and transformation is linear) then y =2x is invariant.		**A1**	2.1
	Notes			
	M1	Identifies a point on the line y = 2x and finds its image under T. If (0,0) is used there must be a clear statement it is because this is on the line, but for other points accept with any line on y = 2x without statement.		
	A1	Shows the image and another point, which may be (0,0), on y=2x both map to points on y = 2x concludes line is invariant. Need not reference transformation being linear for either mark here.		
Alt 3	$\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ mX+c \end{pmatrix} \Rightarrow \begin{matrix} 4x-5(mx+c) = X \\ 2x-7(mx+c) = mX+c \end{matrix}$ $\Rightarrow 2x-7(mx+c) = m\left(4x-5(mx+c)\right)+c$ $\Rightarrow \left(5m^2-11m+2\right)x+\left(5m-8\right)c = 0$ $\Rightarrow (5m-1)(m-2) = 0 \Rightarrow m = \ldots$ Or similar work with c = 0 throughout.		**M1**	2.1
	$\left(5m-8 \neq 0 \Rightarrow c = 0\right)$ Hence m = 2 gives an invariant line (with c = 0), so y = 2x is invariant.		**A1**	1.1b
	Notes			
	M1	Attempts to find the equation of a general invariant line, or general invariant line through the origin (so may have c = 0 throughout). To gain the method mark they must progress from finding the simultaneous equations to forming a quadratic in m and solving to a value of m.		
	A1	Correct quadratic in m found, with m = 2 as solution (ignore the other) and deduction that hence y = 2x is an invariant line. Ignore errors in the (5m – 8) here as c = 0 is always a possible solution. No need to see c = 0 derived.		

Examiner Comments:

Alt 2 was not common, but Alt 3 was fairly common as student's confused the ideas of showing a particular line is invariant as opposed to classifying all invariant lines. This led to needless extra work being done in part (c).

Student Response A

a) $\det M = (-7)(4) - (-5)(2) = -28 - (-10) = 38$

b) $M^{-1} = \frac{1}{38}\begin{pmatrix}-7 & 5 \\ -2 & 4\end{pmatrix}$

Area $= \frac{1}{2} \times \text{base} \times \text{height}$

c) $\begin{pmatrix}4 & -5 \\ 2 & -7\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}x \\ 2x\end{pmatrix}$

$4x - 5y = x \Rightarrow 3x - 5y = 0$

$2x - 7y = 2x \Rightarrow -7y = 0$

1/6

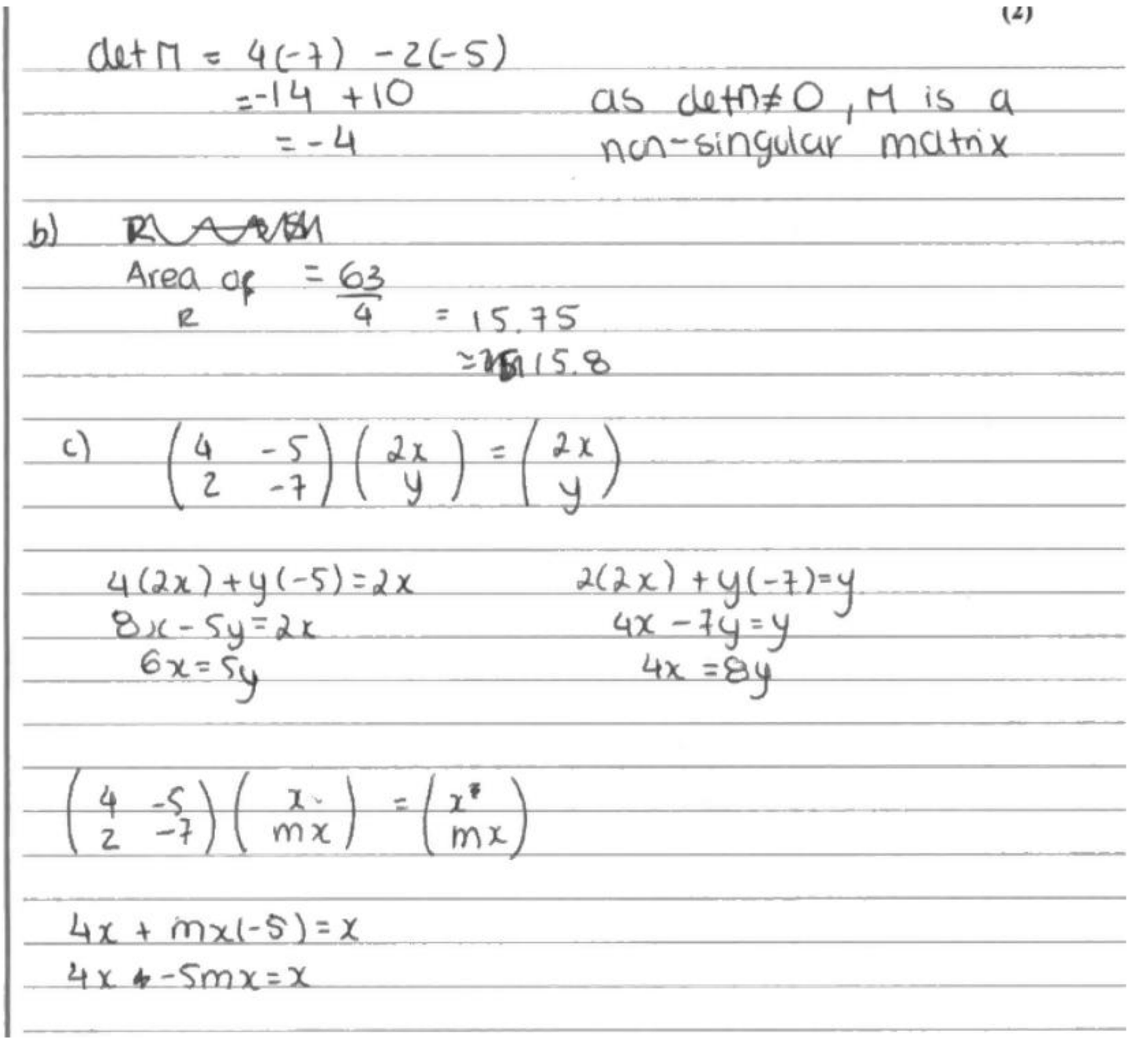
Examiner Comment: (a) M1A0 (b) M0A0 (c) M0A0

An attempt at the determinant is made to gain the first method mark, but the accuracy in (a) is lost because the determinant is incorrectly evaluated. There is also no reason for being singular given nor a conclusion, and each of these aspects would be required for the accuracy.

No attempt to divide the area of *S* by their determinant or modulus thereof is made in part (b) and so the method is not earned.

In part (c) an incorrect method is applied as $y = 2x$ is never applied to the initial point being mapped.

Student Response B


(2)

$\det M = 4(-7) - 2(-5)$
$= -14 + 10$
$= -4$

as $\det M \neq 0$, M is a non-singular matrix

b) Area of R $= \frac{63}{4} = 15.75$
≈ 15.8

c) $\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} 2x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix}$

$4(2x) + y(-5) = 2x$
$8x - 5y = 2x$
$6x = 5y$

$2(2x) + y(-7) = y$
$4x - 7y = y$
$4x = 8y$

$\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x \\ mx \end{pmatrix}$

$4x + mx(-5) = x$
$4x - 5mx = x$

3/6

Examiner Comment: (a) M1A0 (b) M1A1 (c) M0A0

A correct attempt at the determinant is made in part (a) to gain the method mark, but the initial expression is evaluated to an incorrect value of −4 losing the accuracy.

In part (b) the student uses the correct method of dividing by the modulus of their determinant to gain the follow accuracy as well as method (gained for the answer 63/4).

A version of Alt 2 of the scheme is attempted in part (c) but insufficient progress is made to access the method mark. The result of the multiplication should be (X, mX) or similar (not the same (x,mx) as the original point) and they must proceed to a quadratic in m by elimination in order to gain access to the method.

Student Response C

a) $\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}$

$\det M = (4 \times -7) - (-5 \times 2)$
$= -28 - -10$
$= -18$

$\det M \neq 0$ therefore matrix M is non-singular

b) $\begin{pmatrix} 4 & -5 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

~~area = 18~~ $\det M = 18$
area $S = 63$

area of $R = \frac{63}{18} = \frac{7}{2}$ square units

c) $\begin{pmatrix} 4 & -5 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} t \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix}$

$4t - 5y = x$
$2t - 7y = 2x$

~~St~~

$\begin{pmatrix} 4 & -5 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 4x - 10x \\ 2x - 14x \end{pmatrix} = \begin{pmatrix} 2x - 5x \\ 2x - 7 \end{pmatrix}$

hence the line is invariant

5/6

Examiner Comment: (a) M1A1 (b) M1A1 (c) M1A0

Correct and well expressed in part (a).

The correct answer is reached in part (b) and the erroneous "det **M** = 18" comment is interpreted as meaning the modulus of the determinant, as the correct method is carried out.

The correct process of multiplying the matrix by a general point on the line $y = 2x$ is carried out, but accuracy is lost as the result is not correctly simplified to show the image remains on the line.

Exemplar Question 2

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2. The cubic equation

$$2x^3 + 6x^2 - 3x + 12 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are (α +3), (β +3) and (γ + 3), giving your answer in the form $pw^3 + qw^2 + rw + s = 0$, where p, q, r and s are integers to be found.

(5)

(Total for Question 2 is 5 marks)

Mean Score 4.2 out of 5

Examiner Comment

This question tested the relationship linear transformations of roots of polynomial equations (4.2). Full marks were achieved by the majority of students. Although a topic new to the specification, this type of question has been seen numerous times in preparatory material and is a good indicator that students are already adapting well to the new content.

Mark Scheme

Question	Scheme	Marks	AOs
2.	$\{w = x + 3 \Rightarrow\}\ x = w - 3$	B1	3.1a
	$2(w-3)^3 + 6(w-3)^2 - 3(w-3) + 12\ (=0)$	M1	1.1b
	$2w^3 - 18w^2 + 54w - 54 + 6(w^2 - 6w + 9) - 3w + 9 + 12 (=0)$		
	$2w^3 - 12w^2 + 15w + 21 = 0$ (So $p = 2$, $q = -12$, $r = 15$ and $s = 21$)	M1 A1 A1	3.1a 1.1b 1.1b
		(5)	
ALT 1	$\alpha + \beta + \gamma = -\frac{6}{2} = -3,\ \alpha\beta + \beta\gamma + \alpha\gamma = -\frac{3}{2},\ \alpha\beta\gamma = -\frac{12}{2} = -6$	B1	3.1a
	sum roots $= \alpha + 3 + \beta + 3 + \gamma + 3$ $= \alpha + \beta + \gamma + 9 = -3 + 9 = 6$ pair sum $= (\alpha+3)(\beta+3) + (\alpha+3)(\gamma+3) + (\beta+3)(\gamma+3)$ $= \alpha\beta + \alpha\gamma + \beta\gamma + 6(\alpha+\beta+\gamma) + 27$ $= -\frac{3}{2} + 6\times -3 + 27 = \frac{15}{2}$ product $= (\alpha+3)(\beta+3)(\gamma+3)$ $= \alpha\beta\gamma + 3(\alpha\beta + \alpha\gamma + \beta\gamma) + 9(\alpha+\beta+\gamma) + 27$ $= -6 + 3\times -\frac{3}{2} + 9\times -3 + 27 = -\frac{21}{2}$	M1	3.1a
	$w^3 - 6w^2 + \frac{15}{2}w - \left(-\frac{21}{2}\right)\ (=0)$	M1	1.1b
	$2w^3 - 12w^2 + 15w + 21 = 0$ (So $p = 2$, $q = -12$, $r = 15$ and $s = 21$)	A1 A1	1.1b 1.1b
		(5)	
	(5 marks)		

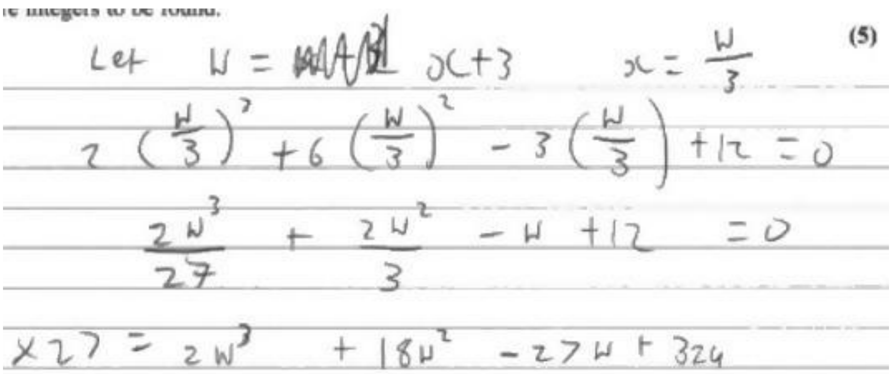
Notes		
	B1	Selects the method of making a connection between x and w by writing $x = w - 3$
	M1	Applies the process of substituting their $x = aw \pm b$ into $2x^3 + 6x^2 - 3x + 12\ (= 0)$ So accept e.g. if $x = \frac{w}{3}$ is used.
	M1	Depends on having attempted substituting either $x = w - 3$ or $x = w + 3$ into the equation. This mark is for manipulating their resulting equation into the form $pw^3 + qw^2 + rw + s(=0)$ ($p \neq 0$). The "= 0" may be implied for this.
See note	A1	At least three of p, q, r and s are correct in an equation with integer coefficients. (need not have "= 0")
	A1	Correct final equation, including "=0". Accept integer multiples.
ALT 1	B1	Selects the method of giving three correct equations each containing α, β and γ.
	M1	Applies the process of finding sum roots, pair sum and product.
	M1	Applies $w^3 - (\text{their sum roots})w^2 + (\text{their pair sum})w - (\text{their product})\ (= 0)$ Must be correct identities, but if quoted allow slips in substitution, but the "=0" may be implied.
See note	A1	At least three of p, q, r and s are correct in an equation with integer coefficients. (need not have "=0")
	A1	Correct final equation, including "=0". Accept multiples with integer coefficients.
Note: may use another variable than w for the first four marks, but the final equation must be in terms of w		
Notes: Do not isw the final two A marks – if subsequent division by 2 occurs then mark the final answer.		

Examiner Comment

Responses to the question this year had a strong bias towards the main scheme, which is simpler to carry out and often more successful. The majority of students who selected this method scored 4 or 5 marks. In contrast students who used the sum/product of roots approach were much more prone to error and likely to score only 3.

In both of these methods, mistakes were seen in the manipulation involved but students demonstrated a sound knowledge of the process to be followed. Omission of the "=0" or use of alternative variables was rare.

Student Response A


(5)

Let $w = x + 3$ $\quad x = \frac{w}{3}$

$2\left(\frac{w}{3}\right)^3 + 6\left(\frac{w}{3}\right)^2 - 3\left(\frac{w}{3}\right) + 12 = 0$

$\frac{2w^3}{27} + \frac{2w^2}{3} - w + 12 = 0$

$\times 27 = 2w^3 + 18w^2 - 27w + 324$

1/5

Examiner Comment: B0M1M0A0A0

Although a correct equation in *x* and *w* is written down, *w* = *x* + 3, this is not correctly rearranged into the required expression for *x* in terms of *w* and so the initial B mark is not awarded. As $x = w/3$ is a linear expression for *x* in terms of *w* the method for substituting into the equation is awarded, but the second method required $x = w \pm 3$ to be used, and so no further marks were available for this response.

Student Response B

$x+3=w$

$x=w-3$

$2(w-3)^3+6(w-3)^2-3(w-3)+12=0$

$2(w^3-9w^2+27w-27)+6(w^2-6w+9)-3(w-3)+12=0$

$2w^3-18w^2+54w-54+6w^2-36w+54-3w+9+12$

$\dfrac{2w^3-12w^2+15w+21}{2}$

$w^3-6w^2+\frac{15}{2}w+\frac{21}{2}$

$(w-3)(w-3)(w-3)$

~~$(w-3)(w^2-6w+27)$~~

~~$w^3-6w^2+27w-3w^2+18w-81$~~

~~$w^3-9w^2+45w-81$~~

$(w-3)(w-3)=w^2-3w-3w+9$

w^2-6w+9

$(w-3)(w^2-6w+9)$

$w^3-6w^2+9w-3w^2+18w-27$

$w^3-9w^2+27w-27$

3/5

Examiner Comment: B1M1M1A0A0

This response follows the main scheme and correctly identifies $x = w - 3$ for the first mark, and proceeds to substitute into the equation and simplify for the two method marks. Although a correct expression is initially reached, the student subsequently divides by 2 and so their final equation does not have integer coefficients and cannot score either accuracy mark. The "=0" is missing so the final A would not have been gained if the erroneous division by 2 had not occurred.

The second page contained rough working only.

Student Response C

(2)

$\alpha_1 = \alpha + 3$

$\beta_1 = \beta + 3$

$\gamma_1 = \gamma + 3$

$\sum \alpha = -\frac{b}{a}$

$\sum \alpha\beta = \frac{c}{a}$

$\sum \alpha\beta\gamma = -\frac{d}{a}$

$\alpha + 3 + \beta + 3 + \gamma + 3 = \alpha + \beta + \gamma + 9 = -\frac{b}{a}$

$\alpha + \beta + \gamma = -\frac{6}{2} = -3 \quad \therefore -\frac{b}{a} = -3 + 9 = 6 \quad \frac{b}{a} = -6$

$(\alpha + 3)(\beta + 3) + (\alpha + 3)(\gamma + 3) + (\beta + 3)(\gamma + 3) =$

$(\alpha\beta + 3\alpha + 3\beta + 9) + (\alpha\gamma + 3\alpha + 3\gamma + 9) + (\beta\gamma + 3\beta + 3\gamma + 9) =$

$\alpha\beta + \alpha\gamma + \beta\gamma + 6\alpha + 6\beta + 6\gamma + 27$

$= \alpha\beta + \alpha\gamma + \beta\gamma + 6(\alpha + \beta + \gamma) + 27$

$= \frac{c}{a} + 6\left(-\frac{b}{a}\right) + 27 = -\frac{3}{2} + 6(6) + 27 = \frac{129}{2}$

$\therefore \frac{c}{a} = \frac{129}{2}$

$(\alpha + 3)(\beta + 3)(\gamma + 3) = (\alpha\beta + 3\alpha + 3\beta + 9)(\gamma + 3)$

$= (\alpha\beta\gamma + 3\alpha\gamma + 3\beta\gamma + 9\gamma + 3\alpha\beta + 9\alpha + 9\beta + 27)$

$= \alpha\beta\gamma + 3(\alpha\beta + \alpha\gamma + \beta\gamma) + 9(\alpha + \beta + \gamma) + 27$

$= -\frac{12}{2} + 3\left(-\frac{3}{2}\right) + 9\left(-\frac{6}{2}\right) + 27 = -\frac{12}{2} - \frac{9}{2} - \frac{54}{2} + \frac{54}{2} = -\frac{21}{2}$

$\therefore -\frac{d}{a} = -\frac{21}{2} \quad \therefore \frac{d}{a} = \frac{21}{2}$

$\frac{b}{a} = -6 \quad \frac{c}{a} = \frac{129}{2} \quad \frac{d}{a} = \frac{21}{2}$

$a = 2 \quad b = -12 \quad c = 129 \quad d = 21 \quad 2x^3 - 12x^2 + 129x + 21$

4/5

Examiner Comment: B1M1M1A1A0

The alternative approach to the scheme has been taken by this student, which is in general more prone to error and longer winded than the main scheme. However in this response the method is carried out correctly with just a slip in the new pair sum (the wrong $-b/a$ is substituted), losing the final accuracy mark. The lack of "=0" in the final equation would also forfeit this mark even if the correct expression had been reached.

The correct sum, pair sum and product of roots are seen embedded within the workings for the new sum etc of roots. Ideally these would be clearly stated first, but in cases like this careful scrutiny of the solution is sometimes needed to ensure all three equations for the initial roots were correct.

Exemplar Question 3

3. Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

(6)

(Total for Question 3 is 6 marks)

Mean Score 3.9 out of 6

Examiner Comment

With proof by induction being a familiar topic, this question was another that was generally answered well. The process required for a proof by induction was shown by most students, with attempts at the base case and an assumption statement seen in almost all cases (though in some the attempt at the base case was insufficient for the first mark). How to carry out the inductive step still proves a challenge for many students, though.

Though many students knew the process of needing to add the $k + 1$ th term to the sum of k terms, there were some who did not manage to achieve this, instead attempting to add the formula for the sum of k and $k + 1$ terms together, or similar. But for those who did set up the correct sum, they generally proceeded at least as far as to reaching the correct expression for the sum to $k+1$ terms. The final accuracy mark was then achieved most of the time, though there were also many instances where it was lost due to an insufficient concluding statement or failing to show the expression in the correct form.

A few students did try to prove the result with the aid of the summation formulae, applied incorrectly to the denominator. Such attempts scored no marks.

Mark Scheme

Question	Scheme	Marks	AOs
3	$n=1,\ \sum_{r=1}^{1}\frac{1}{(2r-1)(2r+1)}=\frac{1}{1\times3}=\frac{1}{3}$ and $\frac{n}{2n+1}=\frac{1}{2\times1+1}=\frac{1}{3}$ (true for n=1)	B1	2.2a
	Assume general statement is true for $n=k$. So assume $\sum_{r=1}^{k}\frac{1}{(2r-1)(2r+1)}=\frac{k}{2k+1}$ is true.	M1	2.4
	$\left(\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\right)\text{"}\frac{k}{2k+1}\text{"}+\frac{1}{(2k+1)(2k+3)}$	M1	2.1
	$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$	dM1	1.1b
	$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}=\frac{(k+1)}{2(k+1)+1}$ or $\frac{k+1}{2k+3}$	A1	1.1b
	As $\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\frac{(k+1)}{2(k+1)+1}$ then the general result is true for $n=k+1$ As the general result has been shown to be true for $n=1$, and true for $n=k$ implies true for $n=k+1$, so the result is true for all $n\in\mathbb{N}$	A1cso	2.4
		(6)	
	(6 marks)		

Notes		
	B1	Substitutes $n=1$ into both sides of the statement to show they are equal. As a minimum expect to see $\frac{1}{1\times 3}$ and $\frac{1}{2+1}$ for the substitutions. (No need to state true for n = 1 for this mark.)
	M1	Assumes (general result) true for $n=k$. (Assume (true for) n = k is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for n = k then ... etc.)
	M1	Attempts to add $(k+1)$th term to their sum of k terms. Must be adding the (k+1)th term but allow slips with the sum.
	dM1	Depends on previous M. Combines their two fractions over a correct common denominator for their fractions, which may be $(2k+1)^2(2k+3)$ (allow a slip in the numerator).
	A1	Correct algebraic work leading to $\frac{(k+1)}{2(k+1)+1}$ or $\frac{k+1}{2k+3}$
	A1	**cso** Depends on all except the **B** mark being scored (but must have an attempt to show the n = 1 case). Demonstrates the expression is the correct for n = k + 1 (both sides must have been seen somewhere) and gives a correct induction statement with **all** three underlined statements (or equivalents) seen at some stage during their solution (so true for n = 1 may be seen at the start). For demonstrating the correct expression, accept giving in the form $\frac{(k+1)}{2(k+1)+1}$, or reaching $\frac{k+1}{2k+3}$ and stating "which is the correct form with n = k + 1" or similar – but some indication is needed. Note: if mixed variables are used in working (r's and k's mixed up) then withhold the final A. Note: If n is used throughout instead of k allow all marks if earned.

Examiner Comments:

Note the final A did not depend on the B mark.

Student Response A

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

prove for $n=1$

$$\sum_{r=1}^{1} \frac{1}{(2r-1)(2r+1)} = \frac{1}{2(1)+1} = \frac{1}{3}$$

~~Assume true for~~
Ambiguous case $n=k$

$$\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$$

Prove for $n=k+1$

$$\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \frac{k+1}{2(k+1)+1}$$

$$\sum_{r=1}^{k+1} \frac{1}{4r^2-1}$$

1/6

Examiner Comment: B0M1M0M0A0A0

Only one side of the base case has been evaluated explicitly (the minimum requirement for the left hand side summation is to see $\frac{1}{1\times 3}$) so the B mark is not gained. There is an assumption statement made "Assume true for ambiguous case $n = k$" – the reference to "ambiguous case" does not lose this mark – so the first method mark is awarded. There is no attempt to add the $(k + 1)$th term to the expression for the sum of the first k terms, so M0M0A0A0 follows.

Student Response B

3. Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$

$\frac{k+1}{2(k+1)+1} = \frac{k+1}{2k+3}$

(6)

Let $n = 1$

$LHS = \frac{1}{(2(1)-1)(2(1)+1)}$ $\quad RHS = \frac{(1)}{2(1)+1}$

$= \frac{1}{3}$

$= \frac{1}{1 \times 3} = \frac{1}{3}$

$\therefore$ LHS = RHS, so true for all natural numbers n, by mathematical induction.

Let $n = k$

$$\sum_{r=1}^{k} \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$$

Let $n = k+1$

$$\sum_{r=1}^{k+1} \frac{1}{(2k-1)(2k+1)}$$

$$= \sum_{r=1}^{k} \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

~~$= \frac{k(2k+3)}{(2k+1)(2k+3)} + \frac{1(2k+1)}{(2k+1)(2k+3)}$~~

~~$= \frac{k+1}{(2k+3)(2k+1)}$~~

$$= \frac{k(2k+3)}{2k+1(2k+3)} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

~~$= \frac{k+1}{2k+1}$~~

$$= k + \frac{1}{(2k+3)(2k+3)}$$

3/6

Examiner Comment: B1M0M1M1A0A0

Both sides of the base case was checked with at least the minimum requirements shown, so the first mark is awarded. There is no induction assumption made following this, but "Let $n = k$" and "Let "$n = k + 1$" are written, implying truth for these cases. As such the first M mark is lost. However, the student does add the (k +1)th term to the sum of the first k terms and correct combines the fractions for the next two method marks. But this is not then simplified correctly to one of the required expressions to gain the first A mark, and so both accuracy marks are lost.

Student Response C

For $n=1$

LHS: $\sum_{r=1}^{1} \frac{1}{(2r-1)(2r+1)} = \frac{1}{3}$

RHS: $\frac{1}{2(1)+1} = \frac{1}{3} =$ LHS therefore true for $n=1$

Assume true for $n=k$ that

$$\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$$

For $n=k+1$

$$\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+3)(2k+1)}$$

~~$\frac{1}{(2k+3)(2k+1)}$~~

$$= \frac{k(2k+3)(2k+1)+2k+1}{(2k+1)(2k+3)(2k+1)}$$

~~$\frac{k(2k+2)(2k)+2k+1}{(2k+1)(2k+2)(2k)}$~~

~~$= \frac{4k^3+4k^2+2k+1}{(2k+1)(2k+2)(2k)}$~~

$$= \frac{k(4k^2+8k+3)+2k+1}{(2k+1)(2k+3)(2k+1)}$$

$$= \frac{4k^3+8k^2+5k+1}{(2k+1)(2k+3)(2k+1)}$$

$$= \frac{(k+1)(2k+1)^2}{(2k+1)^2(2k+3)}$$

$$= \frac{k+1}{2k+3}$$

$$= \frac{k+1}{2(k+1)+1}$$

In desired form $\therefore$ true for $n=k+1$

As it is true for $n=1$, $n=k$ and $n=k+1$ it is true for $n \in \mathbb{N}$

4/6

Examiner Comment: B0M1M1M1A1A0

There is insufficient evidence of the substitution into the left hand side of the base case statement. The minimum required is to see $\frac{1}{1 \times 3}$. So the B mark was not awarded. Note that there has been an attempt to establish the base case, though, so the final A is still potentially accessible despite losing this mark.

The induction assumption is made and the inductive step correctly carried out, reaching the required form for the sum of k +1 terms, gaining M1M1A1. However, the final A is not awarded as the concluding statement is not correct. The conclusion must convey the idea of "if true for $n = k$ then true for $n = k + 1$", but the student here claims instead it is true for $n = k$ and $n = k + 1$.

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Exemplar Question 4

4. The line l has equation

$$\frac{x+2}{1}=\frac{y-5}{-1}=\frac{z-4}{-3}$$

The plane Π has equation

$$\mathbf{r}.(\mathbf{i}-2\mathbf{j}+\mathbf{k})=-7$$

Determine whether the line l intersects Π at a single point, or lies in Π, or is parallel to Π without intersecting it.

(5)

(Total for Question 4 is 5 marks)

Mean Score 3.2 out of 5

Examiner Comment

This question tested a geometrical understanding of the interaction of planes and lines, and was unfamiliar to many students. However, most students managed at least 3 marks on this question, with a third or so achieving full marks.

Despite there being a variety of methods, the most common one was via the main scheme approach, and this was also the most successfully attempted. Mixed approaches (the main scheme and Alt 1) tended to revert back to the main scheme at some point. The second alternative was very rare, and commonly unsuccessfully completed.

The majority of students did achieve the equation of the line in parametric form, though a few managed to mix up the format of the equation. For some, this was as far as they proceeded, not knowing what to do with it. But most students did proceed to perform substitute into the plane equation. Mistakes with signs were quite common at this stage, particularly the loss of the – sign from the −7, but these often still led to a contradiction.

When a correct contradiction was achieved, students generally did not realise that this eliminated two of the three given possibilities and so went on to try and prove the line and plane were parallel using the scalar product of directions. Likewise, many started on this route before reverting to the main scheme when needing to determine if they intersected, rather than simply testing one point once they knew line and plane were parallel. The geometry of the situation was not well understood, but the carrying out of procedures was done well.

Mark Scheme

Question	Scheme	Marks	AOs
4.	$(\mathbf{r} =)\begin{pmatrix}-2+\lambda \\ 5-\lambda \\ 4-3\lambda\end{pmatrix}$ **or** $\begin{pmatrix}-2 \\ 5 \\ 4\end{pmatrix}+\lambda\begin{pmatrix}1 \\ -1 \\ -3\end{pmatrix}$ (oe)	M1	1.1b
	So meet if $\begin{pmatrix}-2+\lambda \\ 5-\lambda \\ 4-3\lambda\end{pmatrix}\bullet\begin{pmatrix}1 \\ -2 \\ 1\end{pmatrix}=-7 \Rightarrow (-2+\lambda)\times 1+(5-\lambda)\times -2+(4-3\lambda)\times 1=-7$	M1 A1	3.1a 1.1b
	$\Rightarrow 0\lambda-8=-7 \Rightarrow -8=-7$ a contradiction so no intersection	A1ft	2.3
	Hence l is parallel to Π but not in it.	A1cso	3.2a
		(5)	
	(5 marks)		

Notes

M1	Forms a parametric form for the line. Allow one slip.
M1	Substitutes into the equation of the plane to an equation in λ. May use Cartesian form of plane to substitute into.
A1	Correct equation in λ
A1ft	Simplifies and derives a contradiction and deduces line and plane do not meet. Follow through in their initial equation in λ so - contradiction so no intersection if λ disappears and constants unequal - line lies in plane if a tautology is arrived at - meet in a point if a solution for λ is found. **But do not allow for incorrect simplification from a correct initial equation in λ** Note that a miscopy/misread of 7 instead of −7 can therefore score a maximum of M1M1A0A1A0.
A1cso	Correct deduction from correct working. This may be seen two separate statements in their working. You may see attempts at showing the line is parallel before/after deducing there is no intersection.

Question	Scheme	Marks	AOs
Alt 1	Note that some may a attempt a mix of the main scheme and Alt 1. Mark under main scheme unless Alt 1 would score higher.		
	$\begin{pmatrix}1\\-1\\-3\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\1\end{pmatrix}=1\times1+(-1)\times(-2)+(-3)\times1=0$	M1	3.1a
	Hence l is parallel to Π	A1	1.1b
	$(-2,5,4)$ on l, but $(1)(-2)+(-2)(5)+1(4)=-8$	M1	1.1b
	$-8\neq-7$ so $(-2,5,4)$ is not on the plane.	A1ft	2.3
	Hence l is (parallel to Π but) not in the plane.	A1cso	3.2a
		(5)	
	(5 marks)		
	Alt 1 Notes		
	M1 Attempts the dot product between the two direction vectors. **A1** Shows dot product is zero and makes the correct deduction that line is parallel to plane. **M1** Finds a point on l and substitutes into the equation of Π (vector or Cartesian) **A1ft** Simplifies and derives a contradiction – follow through their equation, so if arrive at a tautology, they should deduce the line is in the plane. **A1cso** Correct deduction from correct working but may be split across working.		
Alt 2	Attempts to solve $\frac{x+2}{1}=\frac{y-5}{-1}=\frac{z-4}{-3}$ and $x-2y+z=-7$ simultaneously – eliminates one variable for M mark.	M1	3.1a
	e.g. $y=-(x+2)+5=-x+3\Rightarrow x-2(-x+3)+z=-7\Rightarrow 3x+z=-1$ (oe)	A1	1.1b
	Solves reduced equations, e.g. $-3(x+2)=z-4\Rightarrow 3x+z=-2$ and $3x+z=-1\Rightarrow(3x+z)-(3x+z)=-2-(-1)$	M1	1.1b
	$\Rightarrow 0=-1$ a contradiction so no intersection	A1ft	2.3
	Hence l is parallel to Π but not in it.	A1cso	3.2a
		(5)	
	(5 marks)		
	Alt 2 notes		
	M1 Attempts to solve the Cartesian equation of the line and plane, using the plane equation to eliminate one variable for the M. **A1** Correct elimination of their chosen variable. (E.g may see $3-3y+z=-7$ or $-2x-2y-2=-7$ etc) **M1** Solves the reduced equations in two variables… **A1ft** … and derives a contradiction/line and plane do not meet. Follow through their result, so may reach a tautology and deduce lies in plane, or find single solution and deduce meet in a point. **A1cso** Correct deduction from correct working.		

Examiner Comments:

A mix of the main scheme and Alt 1 provided the most common approach, but this generally scored marks by the main scheme, which picked up the four marks, with the Alt 1 part only contributing to the final A if correct. Alt 2 was very rare.

Student Response A

$$l: \quad r = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$ is not a multiple of $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\therefore$ the line is not parallel to the plane

$$\begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = -1 + 2 - 3$$
$$= -2$$

$a \cdot n \neq 0$ $\therefore$ the line does not lie in the plane

As the line is not parallel, nor lies on the plane, it must intersect it at a single point

1/5

Examiner Comment: M1A0M0A0A0

The first mark is gained for the (correct) attempt to parametrise the line in the first line of working. But then the student compares the direction of the line to the direction of the normal to the plane, instead, which is not a correct process to determine if they are parallel or intersect. Three is never an attempt to substitute into the equation of the plane, so no further marks are available via the main scheme. There is also an attempt at the dot product of the direction of the line and the normal to the plane, but the result of this is incorrect. This would have gained the first method for the attempt if it had not already been gained, but since it has, this work is worth no extra marks.

Student Response B

line l : $\begin{pmatrix}-2\\5\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$

plane Π : $r.\begin{pmatrix}1\\-2\\1\end{pmatrix}=-7$

or $x-2y+z=-7$

~~$-2-2(5)+4$~~

~~$=-2-10+4$~~

~~$=-8$~~

~~∴ the line doesn't intersect the plane at a single point~~

$-7=a.n$

$-7=\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}$

general point : $\begin{pmatrix}-2+\lambda\\5-\lambda\\4+3\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=-7$

$-2+\lambda-10+2\lambda+4+3\lambda=-7$

$6\lambda=1$

$\lambda=\frac{1}{6}$

the point $\begin{pmatrix}-11/6\\29/6\\9/2\end{pmatrix}$

∴ it intersects at a point.

If direction vector $\cdot\ \underline{n}=0$ then the line and plane is parallel

$\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=1+2+3$

$=6\neq0$ ∴ Π and l are not parallel.

∴ The line l intersects Π at a single point.

3/5

Examiner Comment: M1A0M1A1A0

An attempt to parametrise the line is made, gaining the first mark. One slip in the parametrisation was permitted for the method. An attempt to substitute into the plane equation is then made for the second method mark. The equation formed is incorrect since due to the error with the -3λ in the third coordinate. However, the equation is simplified to find a value of λ and an appropriate conclusion for their equation is made, gaining the follow through accuracy mark. In this case it is that the line and plane intersect in a point. The final A cannot be scored due to the earlier error.

Student Response C

equation of line:

$\lambda = \frac{x+2}{1} \Rightarrow x = \lambda - 2$

$\lambda = \frac{y-5}{-1} \Rightarrow y = -\lambda + 5$

$\lambda = \frac{z-4}{-3} \Rightarrow z = -3\lambda + 4$

$r = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$

The normal vector $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is ~~parallel~~ perpendicular to the plane so if it is also perpendicular to the line, then the line is parallel to the plane.

~~a.b=~~ $a \cdot n = |a||n| \cos\theta$

$\cos\theta = \frac{a \cdot n}{|a||n|} = \frac{(1\times1)+(-2\times-1)+(-3\times1)}{\sqrt{1^2+1^2+3^2} \times \sqrt{1^2+2^2+1^2}}$

$= \frac{0}{\sqrt{66}} = 0$

so $\cos\theta = 0$

$\theta = \cos^{-1} 0 = 90°$

$\therefore$ the line is parallel to the plane.

Does the line intersect the plane:

$\Pi: x - 2y + z + 7 = 0$

$x = \lambda - 2 \quad y = -\lambda + 5 \quad z = -3\lambda + 4.$

$(\lambda - 2) - 2(-\lambda + 5) + (-3\lambda + 4) + 7 = 0$

$\lambda - 2 + 2\lambda - 10 - 3\lambda + 4 + 7 = 0$

$-1 = 0$

This is not possible ∴ the line does not intersect the plane ∴

The line l is parallel to Π without intersecting it.

5/5

Examiner Comment: M1A1M1A1A1

This response is fully correct but exemplifies the common approach of a mix of method. The line is correctly parametrised for the first mark and then on page 2 the coordinates are substituted into the equation of the plane to score the second M and A marks for a correct equation. The equation is simplified to show a (correct) contradiction for the second A mark, deducing there is no intersection. The student has then already shown the line and plane are parallel and draws the correct conclusion. Note that the initial work showing the line and plane are parallel is not necessary but was not incorrect and would under Alt 2 have scored the first two marks only, with the work showing no intersection being needed to gain more.

Exemplar Question 5

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5.

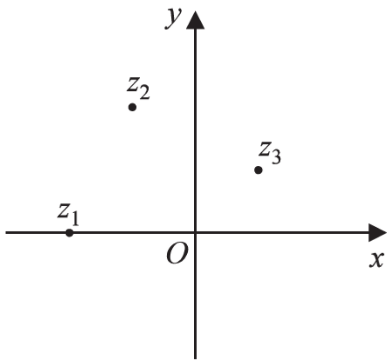


Figure 1

The complex numbers $z_1 = -2$, $z_2 = -1 + 2\text{i}$ and $z_3 = 1 + \text{i}$ are plotted in Figure 1, on an Argand diagram for the complex plane with $z = x + \text{i}y$

(*a*) Explain why z_1, z_2 and z_3 cannot all be roots of a quartic polynomial equation with real coefficients.

(2)

(*b*) Show that $\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \frac{\pi}{4}$

(3)

(*c*) Hence show that $\arctan(2) - \arctan\left(\frac{1}{3}\right) = \left(\frac{\pi}{4}\right)$

(2)

(*d*) Copy Figure 1 and shade the set of points of the complex plane that satisfy the inequality

$$|z + 2| \leq |z - 1 - \text{i}|$$

(2)

(Total for Question 5 is 9 marks)

Mean Score 4.7 out of 9

Examiner Comment

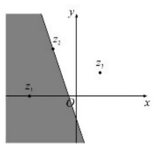
This question tested complex number work from section 2 of the specification. The problem solving aspect of this question proved a challenge for many students, with very few scoring highly on this question. The was a general confusion of what to do between parts (b) and (c), with many not seeing that the calculation via evaluation of the quotient was necessary to prove the result in part (c).

In part (a) the majority of students mentioned complex conjugates, however, very few gave a complete argument. Many failed to mention that a quartic has a maximum of 4 roots or to show that the given values would result in 5 roots.

The main issue with parts (b) and (c) was students confusing the methods. For those who understood what was needed for part (b) the majority correctly worked through multiplying the denominator by the conjugate (or used calculator) to get to $\frac{1}{2}+\frac{1}{2i}$ and achieve the first two marks. Many lost the final A mark due to lack of justification for arctan(1) being $\frac{\pi}{4}$ rather than $-\frac{3\pi}{4}$. Those who did justify largely did so with the use of a diagram. Part (c) was relatively straight forward for those who realised the difference of arguments was needed, and many scored both marks following incorrect attempts at (b).

In part (d) most students scored at least 1 mark here though some did not appreciate that the bisector passed through point z_2. The majority did shade the correct side. Common errors included lines passing through zero and candidates drawing circles. Many attempted this part even if they had become stuck earlier on in the question.

Mark Scheme

Question	Scheme	Marks	AOs
5 (a)	Complex roots of a real polynomial occur in conjugate pairs	M1	1.2
	so a polynomial with z_1, z_2 and z_3 as roots also needs z_2* and z_3* as roots, so 5 roots in total, but a quartic has at most 4 roots, so no quartic can have z_1, z_2 and z_3 as roots.	A1	2.4
		(2)	
(b)	$\dfrac{z_2 - z_1}{z_3 - z_1} = \dfrac{-1+2\text{i}-(-2)}{1+\text{i}-(-2)} = \dfrac{1+2\text{i}}{3+\text{i}} \times \dfrac{3-\text{i}}{3-\text{i}} = \ldots$	M1	1.1b
	$= \dfrac{3-\text{i}+6\text{i}+2}{9+1} = \dfrac{5+5\text{i}}{10} = \dfrac{1}{2}+\dfrac{1}{2}\text{i}$ oe	A1	1.1b
	As $\dfrac{1}{2}+\dfrac{1}{2}\text{i}$ is in the first quadrant (may be shown by diagram), hence $\arg\left(\dfrac{z_2 - z_1}{z_3 - z_1}\right) = \arctan\left(\dfrac{1/2}{1/2}\right)\left(= \arctan(1)\right) = \dfrac{\pi}{4}$ *	A1*	2.1
		(3)	
(c)	$\arg\left(\dfrac{z_2 - z_1}{z_3 - z_1}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_1) = \arg(1+2\text{i}) - \arg(3+\text{i})$	M1	1.1b
	Hence $\arctan(2) - \arctan\left(\dfrac{1}{3}\right) = \dfrac{\pi}{4}$ *	A1*	2.1
		(2)	
(d)	 Line passing through z_2 and the negative imaginary axis drawn.	B1	1.1b
	Area below and left of their line shaded, where the line must have negative gradient passing through negative imaginary axis but need not pass through z_2	B1	1.1b
	Unless otherwise indicated by the student mark Diagram 1(if used) if there are multiple attempts.		
		(2)	
	(9 marks)		

Notes		
(a)	**M1**	Some evidence that complex roots occur as conjugate pairs shown, e.g. stated as in scheme, or e.g. identifying if $-1+2\mathrm{i}$ is a root then so is $-1-2\mathrm{i}$. Mere mention of complex conjugates is sufficient for this mark.
	A1	A complete argument, referencing that a quartic has at most 4 roots, but would need at least 5 for all of z_1, z_2 and z_3 as roots. There should be a clear statement about the number of roots of a quartic (e.g a quartic has four roots), and that this is not enough for the two conjugate pairs and real root.
(b)	**M1**	Substitutes the numbers in expression and attempts multiplication of numerator and denominator by the conjugate of their denominator or uses calculator to find the quotient. (May be implied.) NB Applying the difference of arguments and using decimals is M0 here.
	A1	Obtains $\frac{1}{2}+\frac{1}{2}\mathrm{i}$. (May be from calculator.) Accepted equivalent Cartesian forms.
	A1*	Uses arctan on their quotient and makes reference to first quadrant or draws diagram to show they are in the first quadrant. to justify the argument.
(c)	**M1**	Applies the formula for the argument of a difference of complex numbers and substitutes values (may go directly to arctans if the arguments have already been established). If used in (b) it must be seen or referred to in (c) for this mark to be awarded. Allow for $\arg(z_2-z_1)-\arg(z_3-z_1)$ if z_2-z_1 and z_3-z_1 have been clearly identified in earlier work.
	A1*	Completes the proof clearly by identifying the required arguments and using the result of (b). Use of decimal approximations is A0.
(d)	**B1**	Draws a line through z_2 and passing through negative imaginary axis.
	B1	Correct side of bisector shaded. Allow this mark if the line does not pass through z_2. But it should be an attempt at the perpendicular bisector of the other two points – so have negative gradient and pass through the negative real axis. Ignore any other lines drawn for these two marks.

Student Response A

(2)

z_1, z_2 and z_3 cannot all be roots of a quartic polynomial equation with real coefficients as z_2 and z_3 are both imaginary but one is not the z^* of the other. For example for them to be from the same equation, if $z_2 = -1+2i$ than z_3 would need to be $-1-2i$ but it isnt.

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{(-1+2i)-(-2)}{(1+i)-(-2)} \quad \frac{1+2i}{3+i}$$

$$\frac{(1+2i)(3-i)}{(3+i)(3-i)} = \frac{3-i+6i-2i^2}{9-3i+3i-i^2} = \frac{5+5i}{10}$$

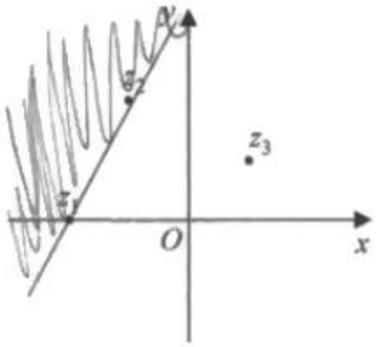


Diagram 1

3/9

Examiner Comment: (a) M1A0 (b) M1A1A0 (c) M0A0 (d) B0B0

Reference to the complex conjugates is made in part (a) but the explanation makes no reference to their being a maximum of 4 roots possible to explain why the presence of a fifth root is a problem, so 1 out of 2 is scored for this part.

The correct expression is formed in (b), with attempting to find the Cartesian form made, and the correct Cartesian form, $\frac{5+5i}{10}$, is reached, scoring the first two marks. No attempt to find the argument of this is made.

There is no attempt at part (c), and a line of positive gradient is drawn in part (d) meaning neither mark is accessible.

Student Response B

$z_1 = -2$

$z_2 = -1 + 2i$

$z_3 = 1 + i$

(a) In order for them to be roots of a polynomial with real coefficients, one of the complex roots must be a complex conjugate of another root of the polynomial, in order for the product / sum of all the roots to be real. (The complex roots must come in complex conjugate pairs).

(b) $\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \frac{\pi}{4}$

Question 5 continued

~~ALTERNATIVE METHOD.~~

~~$\arg z_1 = \pi$~~
~~$\arg z_2 = \alpha$~~
~~$\arg z_3 = \beta$~~

$= \arg(z_2 - z_1) - \arg(z_3 - z_1)$

$z_2 - z_1 = (-1+2i) - (-2)$

$z_2 - z_1 = -1 + 2i + 2 = 1 + 2i$

$z_3 - z_1 = (1+i) - (-2)$

$z_3 - z_1 = 1 + i + 2 = 3 + i$

$\arg(1+2i) = \theta_1$

$\tan\theta_1 = \frac{2}{1}$ $\quad \theta_1 = 1.107148718..\text{ rad}$

$\arg(3+i) = \theta_2$

$\tan\theta_2 = \frac{1}{3}$ $\quad \theta_2 = 0.3217505544..\text{ rad}$

$\arg(1+2i) - \arg(3+i) = \theta_1 - \theta_2$

$\theta_1 - \theta_2 = \frac{\pi}{4}$ radians

(c) $\tan\theta_1 = \frac{2}{1}$

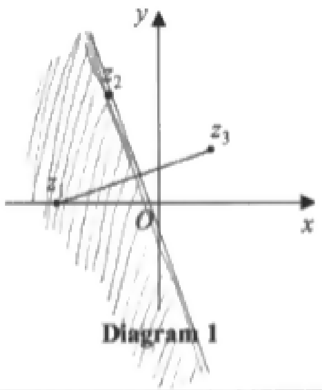
$\theta_1 = \arctan(2)$

$\tan\theta_2 = \frac{1}{3}$

$\arctan\frac{1}{3} = \theta_2$

$\theta_1 - \theta_2 = \frac{\pi}{4})$

$\arctan(2) - \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$



$|z+2| \leq |z-1-i|$

$z_1(-2,0) \quad z_3(1,1)$

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Examiner Comment: (a) M1A0 (b) M0A0A0 (c) M1A1 (d) B1B1

Reference to the complex conjugate is made in part (a) but the explanation makes no reference to their being a maximum of 4 roots possible to explain why the presence of a fifth root is a problem, so 1 out of 2 is scored for this part.

In part (b) there is no attempt to evaluate the quotient required, so the method (and hence accuracy) cannot be gained. Instead the student applies the difference of arguments and attempts a decimal approach, which gains no marks.

In part (c) reference is made to the difference of arguments in part (b) with justification for the arctan(2) and arctan($\frac{1}{3}$) seen. There is no reference to the decimal work in part (c), so the result of part (b) is carried forward and both marks were scored.
A correct diagram with shading is seen in part (d) for both marks.

Student Response C

5.

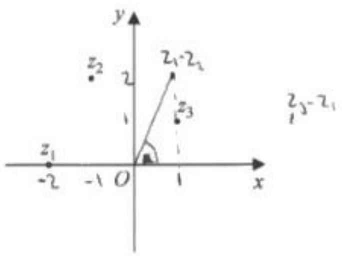


Figure 1

The complex numbers $z_1 = -2$, $z_2 = -1 + 2\mathrm{i}$ and $z_3 = 1 + \mathrm{i}$ are plotted in Figure 1, on an Argand diagram for the complex plane with $z = x + \mathrm{i}y$

(a) Explain why z_1, z_2 and z_3 cannot all be roots of a quartic polynomial equation with real coefficients. **(2)**

(b) Show that $\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \frac{\pi}{4}$ **(3)**

(c) Hence show that $\arctan(2) - \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$ **(2)**

A copy of Figure 1, labelled Diagram 1, is given on page 12.

(d) Shade, on Diagram 1, the set of points of the complex plane that satisfy the inequality

$$|z + 2| \leqslant |z - 1 - \mathrm{i}|$$

(2)

a. if a quartic polynomial had roots z_2 and z_3, its other two roots would be z_2^* and z_3^* so z_1 could not also be a root.

b. $z_2 - z_1 = 1 + 2i = z_4$

$z_3 - z_1 = 3 + i = z_5$

$\arg\left(\frac{z_4}{z_5}\right) = \arg z_4 - \arg z_5$

$\tan\alpha = 2 =$

b. $\frac{(-1+2i)-(-2)}{(1+i)-(-2)} = \frac{1+2i}{3+i}$

$\frac{1+2i}{3+i} \times \frac{3-i}{3-i} = \frac{5+5i}{10} = \frac{1}{2}+\frac{1}{2}i$

$\arg\left(\frac{1}{2}+\frac{1}{2}i\right) = \arctan \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = \arctan(1)$

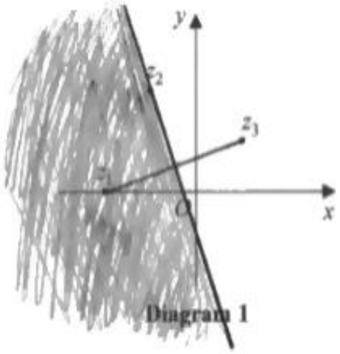
$= \frac{\pi}{4}$

c. $\arg\left(\frac{1+2i}{3+i}\right) = \arg(1+2i) - \arg(3+i)$

$\tan \arg(1+2i) = 2 \Rightarrow \arg(1+2i) = \arctan(2)$

$\tan \arg(3+i) = \frac{1}{3} \Rightarrow \arg(3+i) = \arctan\left(\frac{1}{3}\right)$

$\therefore \arctan(2) - \arctan\left(\frac{1}{3}\right) = \arg\left(\frac{1+2i}{3+i}\right) = \frac{\pi}{4}$



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Examiner Comment: (a) M1A0 (b) M1A1A0 (c) M1A1 (d) B1B1

Reference to the complex conjugate is made in part (a) (by mention of z^* as other roots needed) but the explanation makes no reference to their being a maximum of 4 roots possible to explain why the presence of a fifth root is a problem, so 1 out of 2 is scored for this part.

The correct expression is formed in (b), with attempting to find the Cartesian form made, and the correct Cartesian form is reached, scoring the first two marks. The final mark in (b) is not scored as there is no justification via diagram or reference to quadrants as to why the primary arctangent was required.

The difference of arguments is applied in part (c) with justifications given to establish the result, scoring both marks, and in (d) a fully correct diagram is given.

Exemplar Question 6

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6. An art display consists of an arrangement of n marbles.

When arranged in ascending order of mass, the mass of the first marble is 10 grams.
The mass of each subsequent marble is 3 grams more than the mass of the previous one, so that the rth marble has mass $(7 + 3r)$ grams.

(*a*) Show that the mean mass, in grams, of the marbles in the display is given by

$$\frac{1}{2}(3n + 17)$$

(3)

Given that there are 85 marbles in the display,

(*b*) use the standard summation formulae to find the standard deviation of the mass of the marbles in the display, giving your answer, in grams, to one decimal place.

(6)

(Total for Question 6 is 9 marks)

Mean Score 3.3 out of 9

Examiner Comment

The question tested the summation formulae from section 4.3 of the specification, and was based on some synoptic AS level work on standard deviation to allow students to draw together knowledge from across the whole range of study for maths and further maths.

Part (a) was generally well answered when attempted. Most knew what was required although the overall strategy occasionally got lost and only the sum was found, followed by a disappearance of n with no justification. A small number failed to consider summations at all, whilst others showed a lack of understanding of the need to divide by the number of marbles.

Students found part (b) demanding and fully correct solutions were seen only in a minority of scripts. For some students it was apparent they were answering using the standard deviation formula from the formula booklet, without necessarily understanding what this formula means. This was demonstrated by not realising what $\sum x^2$ represented, as there was no attempt at $\sum(7+3r)^2$, but often they instead attempted to sum the square of the mean, or to use the mean squared.

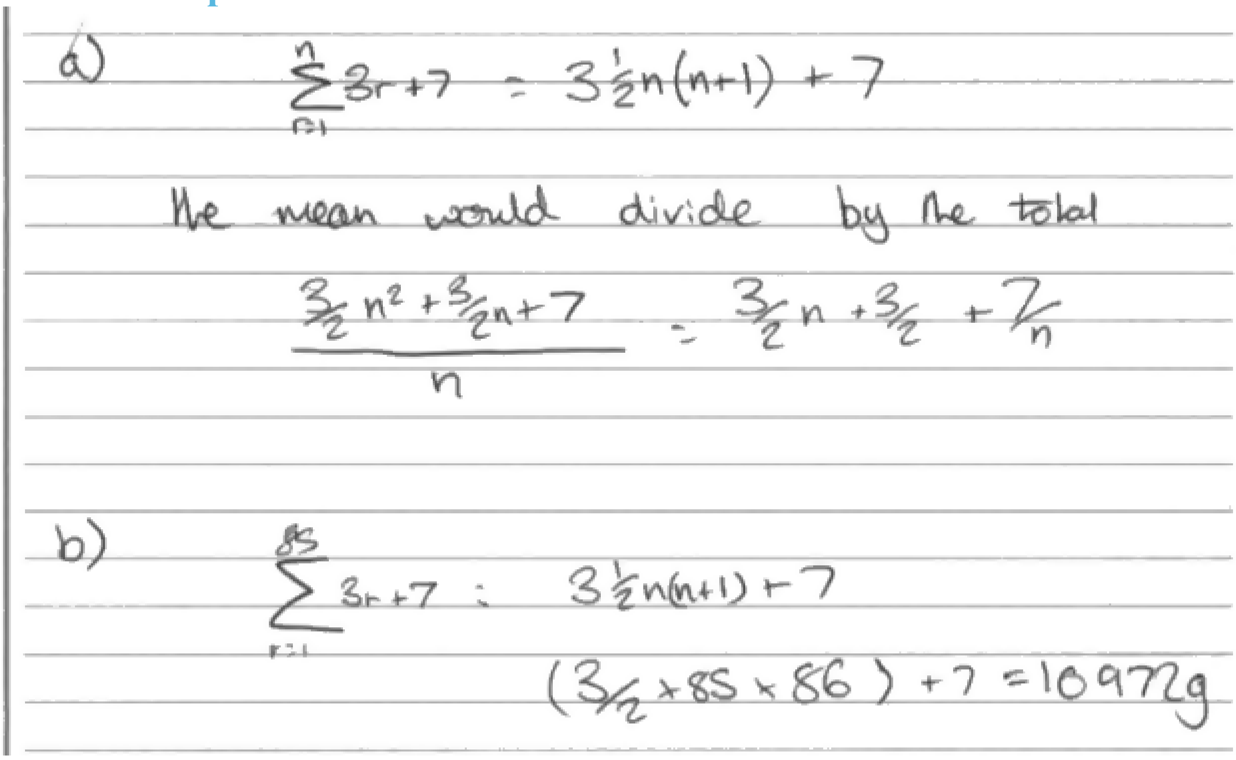
Many students did not realise that they could just substitute into the formula to find the mean. Attempts at variance/standard deviation were very variable in quality. A significant minority of students chose to ignore the question and use the calculator only, rather than use the summation formulae as instructed by the question. These attempts, however, were generally successful in finding the standard deviation, as students attempting the necessary summation generally knew what they needed to find overall.

Mark Scheme

Question	Scheme	Marks	AOs
6(a)	$(\text{mean } = \bar{x} =)\frac{1}{n}\sum_{r=1}^{n}(7+3r)$	M1	1.1a
	$\sum_{r=1}^{n}(7+3r)=\left(7\sum_{r=1}^{n}1+3\sum_{r=1}^{n}r=\right)7n+3\frac{n}{2}(n+1)$	M1	1.1b
	$\bar{x}=7+\frac{3}{2}(n+1)=\frac{14+3n+3}{2}=\frac{1}{2}(3n+17)*$	A1*	2.1
		(3)	
(b)	Correct overall strategy to find the variance or standard deviation. This must include: • An attempt to find the mean • An attempt at $\sum(7+3r)^2$ as part of their formula (however poor, or if stated and followed by a value or if used with incorrect limits). • An attempt at either variance formula with their mean (allow slips in the formula)	M1	3.1a
(Mean)	$\text{mean }(=\bar{x})=136$	B1	1.1b
(Sum)	Way1: $\sum_{r=1}^{n}(7+3r)^2=\sum_{r=1}^{n}(49+42r+9r^2)$ $=\underline{\underline{49n}}+42\times\underline{\frac{1}{2}n(n+1)}+9\times\underline{\frac{1}{6}n(n+1)(2n+1)}$ Way 2: $\sum_{r=1}^{n}(x_i-\bar{x})^2=\sum_{r=1}^{n}(7+3r-\text{"136"})^2=a\sum_{r=1}^{n}r^2+b\sum_{r=1}^{n}r+c\sum_{r=1}^{n}1$ $=9\times\underline{\frac{1}{6}n(n+1)(2n+1)}-\text{"774"}\times\underline{\frac{1}{2}n(n+1)}+\underline{\underline{\text{"16641"}n}}$	<u>M1</u> <u>B1</u>	1.1b 1.1b
(Variance/standard deviation)	Way 1: $=\frac{\text{"2032690"}}{85}-136^2=\ldots$ or $\frac{\text{"2032690"}}{84}-\frac{85}{84}\times136^2=\ldots$ Way 2: $=\frac{\text{"460530"}}{85}=\ldots$ or $\frac{\text{"460530"}}{84}=\ldots$ (using sample standard deviation).	M1	1.1b
	So s.d = $\sqrt{5418}=73.6\,(\text{g})$ Accept 74.0 (g) if sample s.d. used	A1	1.1b
		(6)	
	(9 marks)		

Notes		
(a)	**M1**	Selects the correct procedure for finding the mean ($\bar{x}$), attempting sum and dividing by *n*.
	M1	Splits the sum and applies the formulae for $\sum r$ (accept $7+3\frac{n}{2}(n+1)$ here) Or uses arithmetic series formula $\frac{1}{2}n(a+l)$ with *a* = 10 and *l* an attempt at $7 + 3\times n$, or $\frac{n}{2}(2a+(n-1)d)$ with *a* = 10 and *d* = 3..
	A1*	Correct work proceeding to the answer with an intermediate step shown. **Special case:** Award M0M1A0 for candidates who use $\frac{1}{2}(a+l)$ or equivalent without justification of the division by *n*.
(b)	**M1**	Correct overall strategy to get as far as the variance of marbles in the collection. The attempt at variance should be recognisable (though allow e.g sign slips in the formula for this mark) and an attempt, however poor, at $\sum(7+3r)^2$ must have been made
	B1	Correct value for the mean for 85 marbles (accept as a single fraction, $\frac{272}{2}$). If a student works algebraically until the last step, a correct final answer will imply this mark.
	M1	Expands brackets and applies summation formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to their expression, either in terms of *n* or with *n* = 85 **but must have correct limits**. Allow for obtaining an expression of the correct form for Way 2 if the mean is kept in terms of "*n*". This mark is for correct application of these two summation formula on an attempt at $\sum_{r=1}^{n}(7+3r)^2$ so accept even if this is not part of an attempt at the variance.
	B1	Correct use of $\sum_{r=1}^{n} 1 = n$ in their expression (must be correct limits).
	M1	**Correctly applies** variance or standard deviation formula with *n* = 85, their attempt at $\sum x^2$ (which need not be using 7 + 3*r* or correct limits) and their mean. Accept use of the sample variance/standard deviation is used (dividing by $n-1$) For reference the variance formula is $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)-\bar{x}^2$ where $x_r = 7+3r$ here, or accept for sample variance $\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \left(\frac{1}{n-1}\sum_{i=1}^{n}x_i^2\right)-\frac{n\bar{x}^2}{n-1}$
	A1	Correct standard deviation to 1 decimal place. If sample standard deviation is used, the answer will be 74.0 g to 1 d.p. (74.04…)
	Note: Question specifies use of summation formula and so these must be seen for the 2nd M and 2nd B mark. However, if just 2032690 appears from a calculator all other marks are available.	

Student Response A



a) $\sum_{r=1}^{n} 3r+7 = 3\frac{1}{2}n(n+1)+7$

The mean would divide by the total

$\frac{\frac{3}{2}n^2+\frac{3}{2}n+7}{n} = \frac{3}{2}n+\frac{3}{2}+\frac{7}{n}$

b) $\sum_{r=1}^{85} 3r+7 = 3\frac{1}{2}n(n+1)+7$

$(\frac{3}{2}\times 85\times 86)+7 = 10972$

2/9

Examiner Comment: (a) M1M1A0 (b) M0B0M0B0M0A0

The correct process of attempting the sum and then dividing by n is attempted, scoring the first method. The sum is split and summation formula applied on Σr, which gains the second method, but $\Sigma 1$ is incorrect so the accuracy cannot be gained. The correct result is never achieved in any case.

There is no overall correct strategy applied in part (b), with no attempt at $\sum(7+3r)^2$ made and no attempt at a variance formula. The correct mean is not found as an incorrect formula from (a) is used, so no marks are scored in part (b).

Student Response B

a) $\sum_{r=1}^{n} 7+3r = 7\sum_{r=1}^{n} 1 + 3\sum_{r=1}^{n} r$

$= 7 + 3\left(\frac{1}{2}n(n+1)\right)$

$= 7 + \frac{3}{2}(n)(n+1)$

$= \dfrac{\left(\frac{3}{2}n^2 + \frac{3}{2}n\right)}{n} + 7$

$= \frac{3}{2}n + \frac{3}{2} + 7$

$= \frac{1}{2}(3n+17)$

b) mean $= \frac{1}{2}(3(85)+17)$

$= 136$

$\sigma = \sqrt{\frac{\sum x^2}{n} + \left(\frac{\sum x}{n}\right)^2}$

$= \sqrt{\qquad + 136^2}$

$\sum_{r=1}^{n}(7+3r)^2$

$\sum_{r=1}^{n} 49 + 42r + 9r^2$

$= 49\sum_{r=1}^{n} 1 + 42\sum_{r=1}^{n} r + 9\sum_{r=1}^{n} r^2$

$= 49 + 45\left(\frac{1}{2}(85)(86)\right) + 9\left(\frac{1}{6}(85)(86)(2(85)+1)\right)$

$= 2029539$

$\sigma = \sqrt{2029539 - 136^2}$

$= 1434.58\ldots$

$= 1434.6$ g

5/9

Examiner Comment: (a) M1M1A0 (b) M1B0M1B0M0A0

The correct process of attempting the sum and then dividing by n is attempted with the slip in not dividing all terms by n for the method condoned for the method. The sum is split and summation formula applied on Σr, but the proof is not fully correct due to the incorrect $\Sigma 1$ and failure to divide this by n, so only the two method marks are gained in part (a).

A correct overall strategy is attempted in part (b), with the mean found and an attempt at the standard deviation formula made with an attempt at $\sum(7+3r)^2$. The correct value of 136 for the mean is seen, and the brackets are expanded with attempt at the summation of integers and squares made, gaining the first three marks in part (b). However, the formula for $\Sigma 1$ is not correct, losing the next B mark, and the standard deviation formula, though initially quoted correctly, is not correctly applied and the final M and A are thus not scored.

Student Response C

(a) mean mass $= \dfrac{\sum_{r=1}^{n}(7+3r)}{n}$ = sum of mass / number of marbles

$= \dfrac{3\sum_{r=1}^{n} r + 7\sum_{r=1}^{n} 1}{n}$

$= \dfrac{\frac{3n}{2}(n+1) + 7n}{n}$

$= \frac{3}{2}(n+1) + 7$

$= \frac{3n}{2} + \frac{3}{2} + 7$

$= \frac{3n}{2} + \frac{17}{2}$

$= \frac{1}{2}(3n+17)$

(b) standard deviation $= \sqrt{\dfrac{\sum x^2}{n} - \bar{x}^2}$

variance $= \dfrac{\sum x^2}{n} - \bar{x}^2$

$= \dfrac{\sum_{r=1}^{n}(7+3r)^2}{n} - \left(\frac{1}{2}(3n+17)\right)^2$

$= \dfrac{\sum_{r=1}^{n} 9r^2 + 42r + 49}{n} - \frac{1}{4}(3n+17)^2$

$= \dfrac{9\sum r^2 + 42\sum r + 49\sum 1}{n} - \frac{1}{4}(9n^2 + 102n + 289)$

$= \dfrac{\frac{9n}{6}(n+1)(2n+1) + \frac{42n}{2}(n+1) + 49n}{n} - \frac{9}{4}n^2 - \frac{51}{2}n - \frac{289}{4}$

$= \frac{9}{6}(n+1)(2n+1) + 21(n+1) + 49 - \frac{9}{4}n^2 - \frac{51}{2}n - \frac{289}{4}$

$n = 85$

variance $= 22059 + 1806 + 49 - \frac{25025}{4} - \frac{4335}{2} - \frac{289}{4}$

$= 23914 - 8496$

$= 15418$

Standard deviation $= \sqrt{15418}$

$= 124.2$ grams

7/9

Examiner Comment: (a) M1M1A1 (b) M1B0M1B1M1A0

Part (a) is fully correct, showing the correct process of attempting the sum and then dividing the result by n, with correct summation formula used and no errors seen.

A correct strategy is employed in part (b), but the first B mark is lost because the mean is not seen or implied as correctly evaluated at any point – an error is made when substituting for $\frac{9n^4}{4}$ (the 25025 should be 65025). Consequently the final A mark is also lost, but the method is all shown and correct, so the other marks are all gained.

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Exemplar Question 7

7. $$\mathrm{f}(z) = z^3 - 8z^2 + pz - 24$$

where p is a real constant.

Given that the equation $\mathrm{f}(z) = 0$ has distinct roots

$$\alpha, \beta \text{ and } \left(\alpha + \frac{12}{\alpha} - \beta\right)$$

(a) solve completely the equation $\mathrm{f}(z) = 0$

(6)

(b) Hence find the value of p.

(2)

(Total for Question 7 is 8 marks)

Mean Score 5.2 out of 8

Examiner Comment

This question tested problem solving in relation to the roots of a quadratic (4.1) with complex roots (2.1). It was generally well answered.

In part (a) the required procedure of using the sum of roots to eliminate β was achieved by most students, albeit amongst similar equations for the pair sum and product. Writing out all the information was a common first step, rather than identifying only that which was needed. Only a very small number failed to successfully eliminate β.

Most could then efficiently multiply throughout by α to get a correct equation though some then made algebraic errors rearranging to a quadratic. Solving the quadratic equation was usually performed correctly. Going on to find the third root seemed more troublesome for many candidates, some of whom did not realise that they had found two different roots already in the conjugate pair. The quickest way of finding the third root was to use the sum of roots being equal to 8, but the fact that the product of all three is 24 was more often used. Other students used longer methods for finding the second and third roots, such as use of the pair sum where a quadratic for β was required to be solved, but such methods were more difficult to complete successfully.

For part (b) most students had a correct method for find p, usually using the pair sum of roots, but slips at various points prevented some students from achieving full marks for the question. Also popular as a method was multiplying out the expression $(z - \alpha)(z - \beta)(z - \gamma)$, and given that α and β are conjugates this could be performed quite quickly for the astute student. Use of the factor theorem was the most direct method, but was used infrequently.

Mark Scheme

Question	Scheme	Marks	AOs
7. (a)	$\alpha+\beta+\left(\alpha+\frac{12}{\alpha}-\beta\right)=8$ so $2\alpha+\frac{12}{\alpha}=8$	M1	1.1b
		A1	1.1b
	$\Rightarrow 2\alpha^2-8\alpha+12=0$ or $\alpha^2-4\alpha+6=0$ $\Rightarrow \alpha=\frac{4\pm\sqrt{(-4)^2-4(1)(6)}}{2(1)}$ or $(\alpha-2)^2-4+6=0 \Rightarrow \alpha=...$	M1	1.1b
	$\Rightarrow \alpha=2\pm i\sqrt{2}$ are the two complex roots	A1	1.1b
	A correct full method to find the third root. Common methods are: Sum of roots = 8 $\Rightarrow$ third root $=8-\left(2+i\sqrt{2}\right)-\left(2-i\sqrt{2}\right)=...$ third root $=2+i\sqrt{2}+\frac{12}{2+i\sqrt{2}}-\left(2-i\sqrt{2}\right)=...$ Product of roots = 24 $\Rightarrow$ third root $=\frac{24}{\left(2+i\sqrt{2}\right)\left(2-i\sqrt{2}\right)}=...$ $(z-\alpha)(z-\beta)=z^2-4z+6\Rightarrow f(z)=\left(z^2-4z+6\right)(z-\gamma)\Rightarrow\gamma=...$ (or long division to find third factor).	M1	3.1a
	Hence the roots of $f(z)=0$ are $2\pm i\sqrt{2}$ and 4	A1	1.1b
		(6)	
(b)	E.g. f(4) = 0 $\Rightarrow 4^3-8\times4^2+4p-24=0\Rightarrow p=...$ Or $p=\left(2+i\sqrt{2}\right)\left(2-i\sqrt{2}\right)+4\left(2+i\sqrt{2}\right)+4\left(2-i\sqrt{2}\right)\Rightarrow p=...$ Or $f(z)=(z-4)(z^2-4z+6)\Rightarrow p=...$	M1	3.1a
	$\Rightarrow p=22$ **cso**	A1	1.1b
		(2)	
	(8 marks)		

Notes		
(a)	**M1**	Equates sum of roots to 8 and obtains an equation in just α .
	A1	Obtains a correct equation in α.
	M1	Forms a three term quadratic equation in α and attempts to solve this equation by either completing the square or using the quadratic formula to give $\alpha = ...$
	A1	$\alpha = 2 \pm i\sqrt{2}$
	M1	Any correct method for finding the remaining root. There are various routes possible. See scheme for common ones. Allow this mark if –24 is used as the product. See note below for a less common approach.
	A1	Third root found with all three roots correct. Note α and β need not be identified.
(b)	**M1**	Any correct method of finding p. For example, applies the factor theorem, process of finding the pair sum of roots, or uses the roots to form f(z).
	A1	$p = 22$ by correct solution only. Note: this can be found using only their complex roots from (a) (e.g. by factor theorem)

Note for (a) final M – it is possible to find the second and third roots using only one initial root (e.g. if second root forgotten or error leads to only one initial root being found).

Product of roots = $\alpha\beta\left(\alpha + \frac{12}{\alpha} - \beta\right) = 24 \Rightarrow \alpha\beta^2 - \left(\alpha^2 + 12\right)\beta + 24 = 0$, substitutes in α and attempts to solve the quadratic in β to achieve remaining roots. The final M can be gained once three roots in total have been obtained. (This is unlikely to be seen as part of a correct answer.) Allow if −24 has been used for the product.

Student Response A

(2)

$z^3 - 8z^2 + pz - 24$

$\alpha + \beta + \alpha + \frac{12}{\alpha} - \beta = \frac{-(-8)}{1}$

$2\alpha + \frac{12}{\alpha} = 8$

$2\alpha^2 + 12 - 8\alpha = 0 \qquad \alpha = -3 - \sqrt{13}$

$-3 + \sqrt{13}$.

$\alpha\beta\left(\alpha + \frac{12}{\alpha} - \beta\right)$

$\alpha^2\beta + 12\beta - \beta = -24$

$\alpha^2\beta + 11\beta + 24 = 0$

$22 - 6\sqrt{13}\beta + 11\beta + 24 = 0$

$22 + 6\sqrt{13}\beta + 11\beta + 24 = 0$

1st root $\beta = \frac{11 - 2\sqrt{13}}{8}$ ~~$\alpha = -3 + \sqrt{13}$~~

2nd $\alpha = -3 + \sqrt{13}$.

3rd $= \frac{37 + 34\sqrt{13}}{8}$

b) p = Sum of double

$\frac{p}{1}$ = Sum of double.

$$\sum \alpha\beta = \left(\frac{11-2\sqrt{13}}{8}\right)\left(-3+\sqrt{13}\right) +$$

$$\left(\frac{11-2\sqrt{13}}{8}\right)\left(\frac{37+34\sqrt{13}}{8}\right) +$$

$$\left(-3+\sqrt{13}\right)\left(\frac{37+34\sqrt{13}}{8}\right)$$

$p = 21.81$

3/8

Examiner Comment: (a) M1A1M0A0M0A0 (b) M1A0

The sum of roots is equated to 8 and the equation in just α is produced for the first two marks. Though a correct 3 term quadratic is then formed from this, the student shows no method for solving the quadratic and the answer given are not correct, so the method cannot be awarded (an incorrect quadratic formula is implied, with the roles of "*b*" and "*c*" reversed).

To find the third root the student attempts the method in the note of the mark scheme, using $\alpha\beta\left(\frac{\alpha+12}{\alpha-\beta}\right)=-24$ with one root, but an error in expanding means a quadratic in β is never reached so the method mark for this approach is not available.

An attempt at the pair sum of their roots is made in part (b) to gain the method mark, but the answer is incorrect as the roots are not correct, so the accuracy cannot be awarded.

Student Response B

$\alpha + \beta + (\alpha + \frac{12}{\alpha} - \beta) = -\frac{b}{a} = -\frac{-8}{1}$

$2\alpha + \frac{12}{\alpha} = 8.$

$2\alpha^2 + 12 = 8\alpha$

$2\alpha^2 - 8\alpha + 12 = 0$

$\alpha^2 - 4\alpha + 6 = 0.$

$(\alpha - 2)^2 + 2 = 0$

$\alpha - 2 = \pm 2i$

$\alpha = 2 \pm 2i$

$(z - (2+2i))(z - (2-2i)) = z^2 - (2-2i)z - (2+2i)z + (2+2i)(2-2i)$

$= z^2 - 2z + 2iz - 2z - 2iz + 4 + 4$

$= z^2 - 4z + 8.$

$(z^2 - 4z + 8)(z - a) = z^3 - 8z^2 + pz - 24.$

$-az^2 + 4az - 8a + z^3 - 4z^2 + 8z = z^3 + (-4-a)z^2 + (4a+8)z - 8a.$

$-8a = -24$

$a = 3.$

$\therefore$ p(z) has roots $z = 2+2i,\ 2-2i,\ 3.$

b) $\alpha\beta + \alpha(\alpha + \frac{12}{\alpha} - \beta) + \beta(\alpha + \frac{12}{\alpha} - \beta)$

$\Rightarrow \alpha\beta + \alpha^2 + 12 - \alpha\beta + \alpha\beta + \frac{12\beta}{\alpha} - \beta^2 = p.$

$\Rightarrow (2+2i)(2-2i) + (2+2i)(3) + (2-2i)(3) = p.$

$\Rightarrow 4 + 4 + 6 + 6i + 6 - 6i = p$

$\Rightarrow p = 20.$

5/8

Examiner Comment: (a) M1A1M1A0M1A0 (b) M1A0

The sum of roots is equated to 8 and a correct equation in α is produced to score the first two marks of the question. The student forms a 3 term quadratic from this and attempts to solve using completion of the square, sufficient for the method but a slip when square rooting means the roots are incorrect so the accuracy is lost. A correct method is used to find the final root, forming the quadratic from the two complex roots and then factorising to find the third linear term.

Again in part (b) a correct method is used to find the value of *p*, attempting the pair sum, but the answer is incorrect due to the earlier errors.

Student Response C

a) [crossed out] $-\frac{b}{a} = \alpha + \beta + \left(\alpha + \frac{12}{\alpha} - \beta\right)$

$\frac{c}{a} = \alpha\beta + \alpha\left(\alpha + \frac{12}{\alpha} - \beta\right) + \beta\left(\alpha + \frac{12}{\alpha} - \beta\right)$

$-\frac{d}{a} = \alpha\beta\left(\alpha + \frac{12}{\alpha} - \beta\right)$

$24 = \alpha\beta\left(\alpha + \frac{12}{\alpha} - \beta\right)$

$+8 = \alpha + \beta + \left(\alpha + \frac{12}{\alpha} - \beta\right)$

$\alpha =$ [crossed out] $+8 - \left(\beta + \left(\alpha + \frac{12}{\alpha} - \beta\right)\right)$

$8 - \left(\alpha\beta + \frac{12\beta}{\alpha} - \beta^2\right) = \alpha$

~~$24\left(8 - \left(\alpha\beta + \frac{12\beta}{\alpha} - \beta^2\right)\right)\beta\left(\alpha + \frac{12}{\alpha} - \beta\right)$~~

$\alpha + \frac{12}{\alpha} - \beta = 0$

$\beta = \alpha + \frac{12}{\alpha}$

$8 = \alpha + \alpha + \frac{12}{\alpha} + \alpha + \frac{12}{\alpha} - \beta$

$8 = 3\alpha + \frac{24}{\alpha} -$ [crossed out] $\alpha - \frac{12}{\alpha}$

$2\alpha + \frac{12}{\alpha} = 8$

$2\alpha^2 + 12 = 8\alpha$

$2\alpha^2 - 8\alpha + 12 = 0$

$\alpha = 2 + \sqrt{2}i$

$\alpha = 2 - \sqrt{2}i$

$\beta = \alpha + \frac{12}{\alpha}$

$\beta = 2 + \sqrt{2}i + \frac{12}{2 + \sqrt{2}i}$

$\beta = 2 - \sqrt{2}i + \frac{12}{2 - \sqrt{2}i}$

$2 - \sqrt{2}i + \frac{24 + 12\sqrt{2}i}{4} = \beta \qquad 4\beta = 8 - 4\sqrt{2}i + 24 +$ [illegible]

$4\beta = 32 + 8\sqrt{2}i$

$\beta = 8 + 2\sqrt{2}i$

OR

$\beta = 2 + \sqrt{2}i + \frac{12}{2+\sqrt{2}i}$

$= 2 + \sqrt{2}i + \frac{24 - 12\sqrt{2}i}{4}$

$4\beta = 8 + 4\sqrt{2}i + 24 - 12\sqrt{2}i$

$\beta = 8 - 2\sqrt{2}i$

$2 + \sqrt{2}i + \frac{12}{2+\sqrt{2}i} - 8 + 2\sqrt{2}i$

$8 + 4\sqrt{2}i + 24 - 12\sqrt{2}i - 32 + 8\sqrt{2}i$

$\underline{0}$

~~$(z-(2-\sqrt{2}i))(z-(8+\sqrt{2}i))$~~

$(z-(2-\sqrt{2}i))(z-(2+\sqrt{2}i))$

$z^2 - 2z + z\sqrt{2}i - 2z + 4 - 2\sqrt{2}i - z\sqrt{2}i$

$+ 2\sqrt{2}i - \sqrt{2}i^2$

$-2i^2$

$(z^2 - 4z + 6)(z^3 - 8z^2 + pz - 24)$

$(z + b)$

$p = -4b + 6$

$16 + 6$

$= 22$

$-24 = 6b$

$b = -4$

$p = 22$

6/8

Examiner Comment: (a) M1A1M1A1M0A0 (b) M1A1

The student begins by listing equations for the sum, pair sum and product, not realising only the sum of roots is required for the first stage. There is an incorrect step assuming $\frac{\alpha+12}{\alpha-\beta} = 0$, and using this to find an incorrect expression for β, however as β cancels out when setting the sum of roots to 8 the correct equation in α is reached so the incorrect work is overlooked. The correct complex roots are found from the quadratic, and the first four marks were awarded.

The incorrect expression for β is then used to attempt the remaining root, which is incorrect for the final method and accuracy in part (a).

For part (b) the correct value of *p* is deduced using only the two correct roots found in part (a) but forming the quadratic from the complex roots and an attempt at factorising out, so both marks were awarded for obtaining the correct value.

Exemplar Question 8

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8. A gas company maintains a straight pipeline that passes under a mountain.

The pipeline is modelled as a straight line and one side of the mountain is modelled as a plane.

There are accessways from a control centre to two access points on the pipeline.

Modelling the control centre as the origin O, the two access points on the pipeline have coordinates $P(-300, 400, -150)$ and $Q(300, 300, -50)$, where the units are metres.

(a) Find a vector equation for the line PQ, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where λ is a scalar parameter.

(2)

The equation of the plane modelling the side of the mountain is $2x + 3y - 5z = 300$

The company wants to create a new accessway from this side of the mountain to the pipeline.

The accessway will consist of a tunnel of shortest possible length between the pipeline and the point $(100, k, 100)$ on this side of the mountain, where k is a constant.

(b) Using the model, find

(i) the coordinates of the point at which this tunnel will meet the pipeline,

(ii) the length of this tunnel.

(7)

It is only practical to construct the new accessway if it will be significantly shorter than both of the existing accessways, OP and OQ.

(c) Determine whether the company should build the new accessway.

(2)

(d) Suggest one limitation of the model.

(1)

(Total for Question 8 is 12 marks)

Mean Score 5.6 out of 12

Examiner Comment

This question tested vector algebra, which continues to be difficult topic for students, especially when involved in a question in context.

In part (a) most knew how to form the equation of a line from 2 points, though there were occasionally some mistakes in calculations and some students neglected to start their equation properly with **r** =… as stated in the question.

Part (b) presented the most difficulty for students with many assuming they should use a line through *M* normal to the plane to find the shortest distance. Some tried to use the distance from a point to a plane formula. Another mistake was to misunderstand where the right angle needs to be when calculating the shortest distance, thus taking an incorrect scalar product. Sketching the situation is advisable as a diagram in such a question can make it clear what needs to be done.

Problems even arose in what should have been the straightforward task of using the equation of the plane to find k, which was often badly done with poor algebra in places.

The correct method using the scalar product with the direction of the line was used by a good number of students though, but even amongst such cases, going on to find the coordinate of *M* was quite rare. Instead many found *MX* directly and went on to find the shortest distance required for part (ii). Students ought to check carefully what is asked for, to make sure they answer the question posed. However, many students using an incorrect method to find a value of λ did substitute back into the equation of the line in an attempt to find *M*, so although they had not identified the correct method, they did know what they were supposed to be finding. Many would also then achieve the next method mark for attempting the length *MX*.

For part (c) the correct distances $|OP|$ and $|OQ|$ were usually seen here but many did not give both a comparative reason and conclusion for the accuracy mark. Often, they did not refer to the significantly shorter requirement of the tunnel, assuming any amount shorter would suffice.

Part (d), most students who answered this did give a valid limitation of the model regardless of progress through the previous parts of the question. The most common limitations given were the pipelines or tunnel not being straight lines and flat planes unlikely. Reference simply to inaccurate measurements, however, were not accepted as this is not a limitation of the model as minor inaccuracies in measurement would not affect the interpretation from the model in its context.

Mark Scheme

Question	Scheme	Marks	AOs
8(a)	**Note:** Allow alternative vector forms throughout, e.g row vectors, **i**, **j**, **k** notation $\mathbf{b} = \pm\left[\begin{pmatrix}300\\300\\-50\end{pmatrix} - \begin{pmatrix}-300\\400\\-150\end{pmatrix}\right] = \pm\begin{pmatrix}600\\-100\\100\end{pmatrix}$	M1	1.1b
	So $\mathbf{r} = \begin{pmatrix}-300\\400\\-150\end{pmatrix} + \lambda\begin{pmatrix}600\\-100\\100\end{pmatrix}$ oe $\left(\text{e.g. } \mathbf{r} = \begin{pmatrix}300\\300\\-50\end{pmatrix} + \lambda\begin{pmatrix}6\\-1\\1\end{pmatrix}\right)$	A1	2.5
		(2)	
(b)(i)	$k = 200$	B1	2.2a
	If M is the point on mountain, and X a general point on the line then eg. $\overrightarrow{MX} = \begin{pmatrix}-300\\400\\-150\end{pmatrix} + \lambda\begin{pmatrix}600\\-100\\100\end{pmatrix} - \begin{pmatrix}100\\k\\100\end{pmatrix} = \begin{pmatrix}-400+600\lambda\\400-k-100\lambda\\-250+100\lambda\end{pmatrix} = \begin{pmatrix}-400+600\lambda\\200-100\lambda\\-250+100\lambda\end{pmatrix}$ May be in terms of k or with $k = 200$ used.	M1	3.1b
	e.g. $\begin{pmatrix}-400+600\lambda\\200-100\lambda\\-250+100\lambda\end{pmatrix} \bullet \begin{pmatrix}600\\-100\\100\end{pmatrix} = 0 \Rightarrow \lambda = ...$	dM1	1.1b
	So e.g. $\overrightarrow{OX} = \begin{pmatrix}-300\\400\\-150\end{pmatrix} + \frac{3}{4}\begin{pmatrix}600\\-100\\100\end{pmatrix} = ...$	M1	3.4
	So coordinates of X are $(150, 325, -75)$ Accept as $\begin{pmatrix}150\\325\\-75\end{pmatrix}$	A1	1.1b
		(5)	
(ii)	Length of tunnel is $\sqrt{(150-100)^2 + (325-200)^2 + (-75-100)^2} = ...$	M1	1.1b
	Awrt 221m from correct working, so λ must have been correct. (Must include units)	A1	1.1b
		(2)	
(c)	$\left\|\overrightarrow{OP}\right\| = \sqrt{(-300)^2 + 400^2 + (-150)^2} \approx 522$ $\left\|\overrightarrow{OQ}\right\| = \sqrt{300^2 + 300^2 + 50^2} \approx 427$	M1	1.1b
	New tunnel length is significantly shorter than these values so it is likely that the company will decide to build the accessway. Reason and conclusion needed.	A1ft	2.2b
		(2)	
(d)	E.g. The mountainside is not likely to be flat so a plane may not be a good model. The tunnel and/or pipeline will not have negligible thickness so modelling as lines may not be appropriate. A shortest length tunnel may not be possible, or most practical, as the strata of the rock in the mountain have not been considered by the model.	B1	3.5b
		(1)	
		(12 marks)	

Notes		
(a)	**M1**	Attempts the direction between positions *P* and *Q*. If no method shown, two correct entries imply the method.
	A1	A correct equation in the correct form. Any point on the line may used, and any non-zero multiple of the direction. Must begin **r** = …
(b)		**Note:** mark part (b) as a whole.
(i)	**B1**	Correct value of *k* deduced.
	M1	Realises the need to find the distance from the point on the mountain to a general point on the line.
	dM1	Takes the dot product with the direction vector of line and sets to zero and proceeds to find a value of λ. If working with *k* as well, allow for finding either λ in terms of *k* or *k* in terms of λ.
	M1	Substitutes their λ into their line equation. (This may not have come from correct work, but the method is for using the line equation here.) May be implied by two out of three correct coordinates for their λ **Note:** May omit this step and substitute λ into $\overrightarrow{MX}$. This gains M0 here, but can gain M1A1 in (ii) for finding the length of $\overrightarrow{MX}$.
	A1	Correct point.
(b)(ii)	**M1**	Uses the distance formula with their point and *M*, or with their $\overrightarrow{MX}$ from (i). (May be implied by two out of three correct coordinates for their λ)
	A1	Correct distance, including units. Accept awrt 221 m or $25\sqrt{78}$ m
(c)	**M1**	Calculates the two distances *OP* and *OQ*.
	A1ft	Makes an appropriate conclusion for their tunnel length, but distances *OP* and *OQ* must be correct. A reason and a conclusion is needed. Accept for reason e.g "significantly shorter" or "tunnel is more than 100m less than either existing accessway", as these act as a comparative judgement. But do not accept just "shorter" or just inequalities given with no comparative evidence.
(d)	**B1**	Any appropriate criticism of the model given. The model must be referred to in some way – e.g. criticise the straightness/thickness of line, flatness of plane or lack of taking strata etc of mountain into account (as e.g this means line may not be straight). Note: reference to measurements not being correct is **NOT** a limitation of the model.

<u>For reference</u> Some of the other common equations/values of λ in (b)(i) are:

$$\overrightarrow{MX} = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix} = \begin{pmatrix} -400+6\lambda \\ 200-\lambda \\ -250+\lambda \end{pmatrix} \Rightarrow \lambda = 75$$

$$\overrightarrow{MX} = \begin{pmatrix} 300 \\ 300 \\ -50 \end{pmatrix} + \lambda \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix} - \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 200+600\lambda \\ 100-100\lambda \\ -150+100\lambda \end{pmatrix} \Rightarrow \lambda = -\frac{1}{4}$$

$$\overrightarrow{MX} = \begin{pmatrix} 300 \\ 300 \\ -50 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 200+6\lambda \\ 100-\lambda \\ -150+\lambda \end{pmatrix} \Rightarrow \lambda = -25$$

(If the negative direction vectors are used in any case, the value of λ is just the negative of the above.)

Alternatives to 8(b)

Note that variations may occur with the line equation chosen in part (a), but mark as follows:

Question	Scheme	Marks	AOs
Alt 1 (b)(i)	As per main scheme.	B1 M1	2.2a 3.1b
	$d^2 = (-400+600\lambda)^2 + (200-100\lambda)^2 + (-250+100\lambda)^2$ $= 380000\lambda^2 - 570000\lambda + 262500$ $=380000\left(\lambda - \frac{3}{4}\right)^2 + 48750 \Rightarrow \lambda = ...$	dM1	1.1b
	As per main scheme.	M1 A1	3.4 1.1b
		(5)	
(ii)	Length of tunnel is $\sqrt{"48750"} = ...$	M1	1.1b
	Awrt 221m from correct working, so completion of square must have been correct. (Must include units)	A1	1.1b
		(2)	
	Notes		
(i) **B1M1**	As per main scheme.		
M1	Realises the need to find the distance from the point on the mountain to a general point on the line.		
dM1	Attempts the distance or distance squared of $\overrightarrow{MX}$, expands and completes the square to find the value of λ for which distance is minimum. May obtain other forms for the completed square. Look for $A(B\lambda - C)^2 - D + "262500"$ where $A, B, C, D \neq 0$ but B may be 1.		
M1A1	As per main scheme.		
(ii) **M1**	Correct method for the distance. May be as per main scheme, or via extracting from the completed square constant term.		
A1	Correct distance, including units. Accept awrt 221 m or $25\sqrt{78}$ m		
Alt 2 (b)(i)	As per main scheme.	B1 M1	2.2a 3.1b
	$d^2 = (-400+600\lambda)^2 + (200-100\lambda)^2 + (-250+100\lambda)^2$ $= 380000\lambda^2 - 570000\lambda + 262500$ $\frac{\text{d}}{\text{d}x}(d^2) = 0 \Rightarrow 760000\lambda - 570000 = 0 \Rightarrow \lambda = ...$	dM1	1.1b
	As per main scheme.	M1 A1	3.4 1.1b
		(5)	
(ii)	Length of tunnel is $\sqrt{(150-100)^2 + (325-200)^2 + (-75-100)^2} = ...$	M1	1.1b
	Awrt 221m from correct working, differentiation etc must have been correct. (Must include units)	A1	1.1b
		(2)	

Examiner Comments:

An appendix of some uncommon alternative methods for solving (b) were including at the back of the marking scheme.

Student Response A

(1)

a) $\overrightarrow{PQ} = \begin{pmatrix} 300 \\ 300 \\ -50 \end{pmatrix} - \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} = \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix}$

$r = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \lambda \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix}$

b) $2x + 3y - 5z = 300$

$r \cdot \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} = 300$ plane

i. $\begin{pmatrix} -300 + 600\lambda \\ 400 - 100\lambda \\ -150 + 100\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} = 300$

$2(-300 + 600\lambda) + 3(400 - 100\lambda) - 5(-150 + 100\lambda) = 300$

$-600 + 1200\lambda + 1200 - 300\lambda + 300 - 500\lambda = 300$

$400\lambda + 900 = 300$

$400\lambda = \frac{300}{900} = \frac{1}{3}$

$\lambda = \frac{1}{1200}$

$\begin{pmatrix} -300 + \frac{1}{2} \\ 400 - \frac{1}{12} \\ -150 + \frac{1}{12} \end{pmatrix} = \begin{pmatrix} -299.5 \\ \frac{4799}{12} \\ -\frac{1799}{12} \end{pmatrix} \quad -\frac{599}{2}$

coordinates of intersection $= \left(-\frac{5949}{2}, \frac{4799}{12}, -\frac{1799}{12} \right)$

ii. $2x + 3y - 5z = 300$

$$\mathbf{r}\cdot\begin{pmatrix}2\\3\\-5\end{pmatrix} = 300$$

$$\mathbf{r} = \begin{pmatrix}-300\\400\\-150\end{pmatrix} + \lambda\begin{pmatrix}600\\-100\\100\end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix}100\\k\\100\end{pmatrix} + \mu\Big($$

c)

d) Measurements may be inaccurate so real life model

3/12

Examiner Comment: (a) M1A1 (b)(i) B0M0M0M1A0 (ii) M0A0 (c) M0A0 (d) B0

Part (a) is fully correct, including the **r** =.. and as such scores both marks.

In part (b)(i) no attempt is ever made to find *k* or to find *MX*, so the B mark and first two method marks are not scored (the second method being dependent on the first). A value of λ is produced from substituting the general coordinates of a point on the line into the plane, and this is then substituted into the line equation. Since the value of λ did not need to come from correct work, the third method mark in (b) was thus awarded. There is no further correct work in part (b).

Part (c) was omitted, so could not score, and the reason given in part (b) is not a limitation of the model so does not score the mark. A reference to some aspect of the model was needed for this mark.

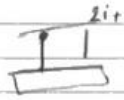
Student Response B

(1)

(a) $\vec{PQ} = \begin{pmatrix}300\\300\\-50\end{pmatrix} - \begin{pmatrix}-300\\400\\-150\end{pmatrix} = \begin{pmatrix}600\\-100\\100\end{pmatrix}$

$r = \begin{pmatrix}-300\\400\\-150\end{pmatrix} + \lambda\begin{pmatrix}600\\-100\\100\end{pmatrix}$

Question 8 continued



$r = \begin{pmatrix}-300+600\lambda\\400-100\lambda\\-150+100\lambda\end{pmatrix}$

$\vec{MP} = \begin{pmatrix}-300+600\lambda\\400-100\lambda\\-150+100\lambda\end{pmatrix} - \begin{pmatrix}100\\k\\100\end{pmatrix} = \begin{pmatrix}-400+600\lambda\\400-k-100\lambda\\-250+100\lambda\end{pmatrix}$

$\begin{pmatrix}~~600~~\\-100\\100\end{pmatrix}$ $\begin{pmatrix}-400+600\lambda\\400-k-100\lambda\\-250+100\lambda\end{pmatrix} \cdot \begin{pmatrix}600\\-100\\100\end{pmatrix} = 0$

$= -240000 + 3600\lambda - 400 + 100k$

$\begin{pmatrix}600\\-100\\100\end{pmatrix} \cdot \Big(\quad -1215400 - 13750 + 100k$

$\begin{pmatrix}-300+600\lambda\\400-100\lambda\\-150+100\lambda\end{pmatrix} \cdot \begin{pmatrix}2\\3\\-5\end{pmatrix} = 0$

$-600 + 1200\lambda + 1200 - 300\lambda + 750 - 500\lambda$

$400\lambda + 1350 = 0 \rightarrow \lambda = -\frac{27}{8}$

$r = \begin{pmatrix}-2325\\737.5\\-487.5\end{pmatrix}$

(c) $\sqrt{(300)^2+(400)^2+(150)^2} = 50\sqrt{109} = 522.01...$

$\sqrt{300^2+300^2+(-50)^2} = 50\sqrt{73} = 427.2...$

∴ Yes it should

(d) pipes may not be going in a straight line

b(ii) $\begin{pmatrix} -400+600\left(-\frac{27}{8}\right) \\ \\ \end{pmatrix}$

$= \begin{pmatrix} -2425 \\ 737.5-k \\ -587.5 \end{pmatrix}$

$\sqrt{2425^2+587.5^2}$

6/12

Examiner Comment: (a) M1A1 (b)(i) B0M1M0M1A0 (ii) M0A0 (c) M1A0 (d) B1

Part (a) is fully correct, including the **r** =.. and as such scores both marks.

In part (b) no value for *k* is ever found, so the B mark is not gained. An attempt at *MX* in terms of *k* is made, however, and this is sufficient for the first method mark. Although the correct dot product is attempted, the student does not proceed to find λ in terms of *k* or vice versa and so the second method mark cannot be awarded. A second attempt with an incorrect dot product is then attempted, which yields an incorrect value of λ, but as this value of λ is substituted into the equation of the line, the third method mark is earned, but the coordinates are not correct as λ is incorrect, so the accuracy is not earned.

Part (b)(ii) is completed after part (d), but gains no marks as the attempt at Pythagoras theorem is only applied to two of the coordinates.

Both *OP* and *OQ* are calculated in part (c), gaining the method mark, but without an answer to (b)(ii) is is impossible to score the accuracy mark in this part as no comparison can be made.
A correct limitation of the model is given in part (d), gaining the mark.

Student Response C

(1)

a) $\overrightarrow{PQ} = \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix}$ $\frac{1}{100}\overrightarrow{PQ} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$

$r = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \lambda\begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$

b)i) ~~$-300 + 6\lambda$~~ ~~$400 - \lambda$~~ ~~$-150 + \lambda$~~

$2(100) + 3k - 5(100) = 300$

$200 + 3k - 500 = 300$

$3k = 600$

$k = 200$

$(100, 200, 100)$

Question 8 continued

ii) ~~$-300 + 6\lambda = 100$~~

~~$6\lambda = 400$~~

~~$\lambda = \frac{200}{3}$~~

~~point on the line~~

$$\begin{pmatrix} -300 + 6\lambda \\ 400 - \lambda \\ -150 + \lambda \end{pmatrix} - \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix} = \begin{pmatrix} -400 + 6\lambda \\ 200 - \lambda \\ -250 + \lambda \end{pmatrix}$$

$$\begin{pmatrix} -400 + 6\lambda \\ 200 - \lambda \\ -250 + \lambda \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$-2400 + 36\lambda - 200 + \lambda - 250 + \lambda = 0$$

$$-2850 + 38\lambda = 0$$

$$\lambda = \frac{2850}{38}$$

$$\lambda = 75$$

$$\left| \begin{matrix} -300 + 6(75) \\ 400 - 75 \\ -150 + 75 \end{matrix} \right| = \sqrt{(150)^2 + (325)^2 + (75)^2}$$

$$= 365.72\text{m}$$

c) $|\vec{OP}| = \sqrt{300^2 + 400^2 + 150^2} = 522.02\text{m}$

$|\vec{OQ}| = \sqrt{(300)^2 + 300^2 + 50^2} = 427.20\text{m}$

~~It is~~ Yes they should as it is ~~almost~~ around 30% ~~smaller~~ shorter than OP and around 14% ~~short~~ shorter than OQ

d) ~~Doesn't take into ~~acco~~ account~~ Unlikely that they will build a perfectly straight accessway

9/12

Examiner Comment: (a) M1A1 (b)(i) B1M1M1M1A0 (ii) M0A0 (c) M1A1 (d) B1

Part (a) is fully correct, with a scaled down direction vector. The **r** = .. is included so both marks are scored.

Good progress is made in part (b) with the value of *k* correctly determined at the start, and a correct process of find vector *MX* and applying the dot product with the direction of the line to find λ. The value of λ is substituted into the line equation in the last line of working for (i) as part of the working, but the correct coordinates are never explicitly found and stated, so the A mark in (b)(i) is forfeited. No marks are scored in (b)(ii) as the student is finding the length of O*X* instead of *MX*.

In part (c) both *OP* and *OQ* are correctly calculated and compared to their *MX* (the answer to (b)(ii) is treated as their *MX* in this case) with a comparative judgement made, and a suitable conclusion drawn, so the follow through accuracy is awarded, scoring both marks for this part.

A correct limitation of the model is given in (d).

Exemplar Question 9

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9. $$f(x) = 2x^{\frac{1}{3}} + x^{-\frac{2}{3}}, \qquad x > 0$$

The finite region bounded by the curve $y = f(x)$, the line $x = \frac{1}{8}$, the x-axis and the line $x = 8$ is rotated through θ radians about the x-axis to form a solid of revolution.

Given that the volume of the solid formed is $\frac{461}{2}$ units cubed, use algebraic integration to find the angle θ through which the region is rotated.

(8)

(Total for Question 9 is 8 marks)

Mean Score 4.5 out of 8

Examiner Comment

This question, testing volume of revolution work (5.1), was generally answered well, though there was a significant minority who did not offer any attempt at all. Most candidates realised that they would need to find the volume of revolution and then scale it to find the angle but had difficulty identifying the correct scaling.

A few did not realise they needed to find y^2 and consequently made little progress. Others did attempt the squaring, but with poor algebra, resulting in only two terms, or incorrect powers, though these were still able to gain the marks for the overall strategy and some for attempting the integration. Most, however, did manage to accurately expand and integrate.

Integration of fractional indices was generally very well done, even if the original expansion was not correct, and most would attempt the correct substitution of limits before attempting scaling. Use of a calculator to evaluate the integral was fortunately rare, as it was costly given that the question specified algebraic integration must be used.

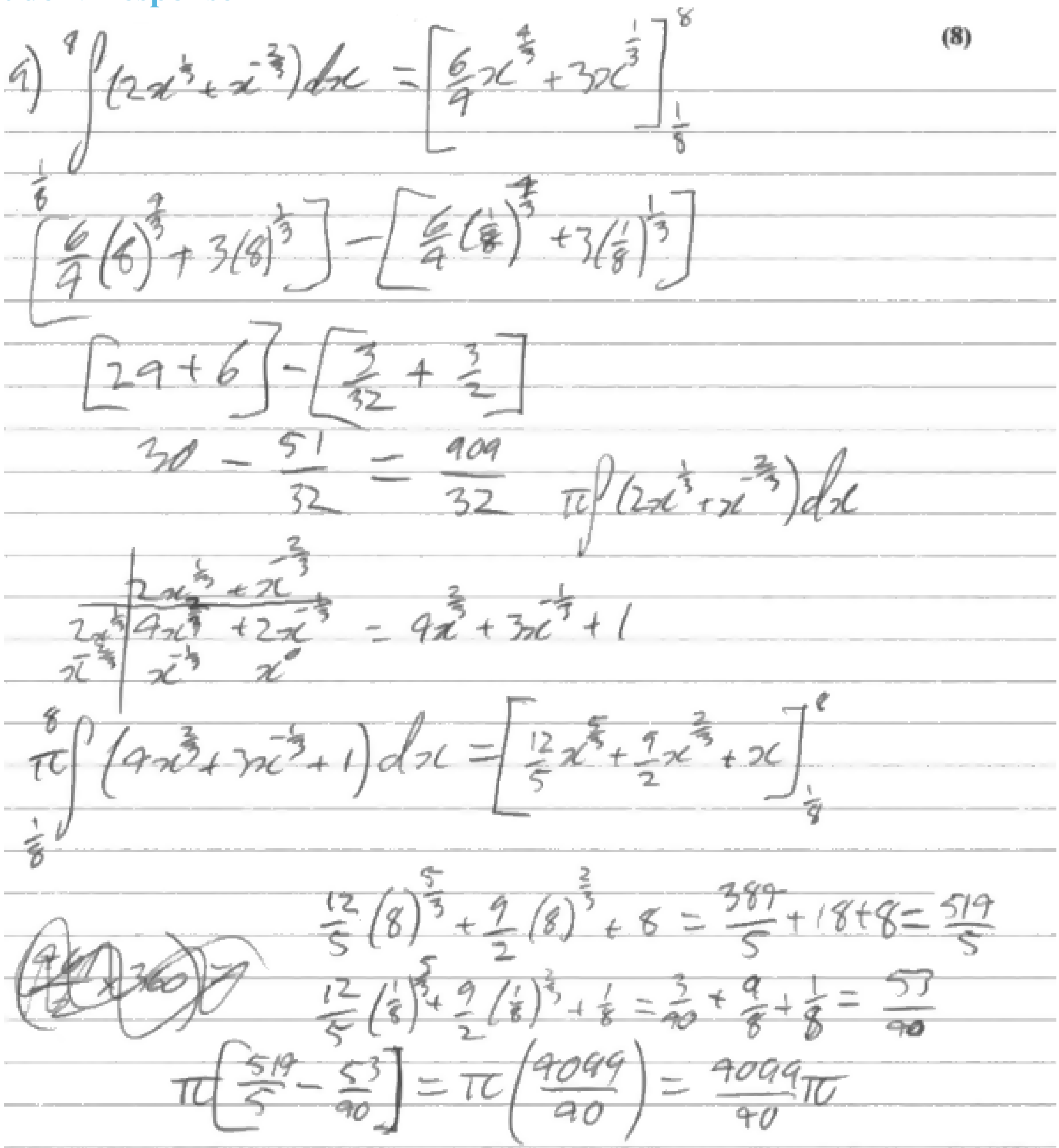
The final two marks were useful discriminator marks, requiring a fully correct strategy to find the required angle. Various incorrect scaling attempts were made. It was more common to first find the volume of rotation through 2π before attempting to scale this volume in some way.

Mark Scheme

Question	Scheme	Marks	AOs
9.	A correct overall strategy, an attempt at integrating y^2 with respect to x combine in some way with the volume of revolution formula (use of $\pi\int y^2\mathrm{d}x$ or $\alpha\int y^2\mathrm{d}x$ for any variable α is fine) followed by attempt to find an angle/form an equation in θ	M1	3.1a
	$y^2 = kx^{\frac{2}{3}} + \ldots + \frac{m}{x^{\frac{4}{3}}}$ or $y^2 = kx^{\frac{2}{3}} + \ldots + mx^{-\frac{4}{3}}$ where … is one or two more terms.	M1	1.1b
	$y^2 = 4x^{\frac{2}{3}} + 4x^{-\frac{1}{3}} + x^{-\frac{4}{3}}$ or $y^2 = 4x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} + x^{-\frac{4}{3}} + 2x^{-\frac{1}{3}}$ (oe)	A1	1.1b
	$\int y^2\mathrm{d}x = \int 4x^{\frac{2}{3}} + \frac{4}{x^{\frac{1}{3}}} + \frac{1}{x^{\frac{4}{3}}}\mathrm{d}x = \alpha x^{\frac{5}{3}} + \beta x^{\frac{2}{3}} + \gamma x^{-\frac{1}{3}}$	M1	1.1b
	$= \frac{12x^{\frac{5}{3}}}{5} + 6x^{\frac{2}{3}} - \frac{3}{x^{\frac{1}{3}}}$ (oe)	A1ft A1	1.1b 1.1b
	$\frac{\theta}{2}\left[\frac{12x^{\frac{5}{3}}}{5} + 6x^{\frac{2}{3}} - \frac{3}{x^{\frac{1}{3}}}\right]_{\frac{1}{8}}^{8} = \frac{461}{2}$ $\Rightarrow \frac{\theta}{2}\left[\left(\frac{12\times 8^{\frac{5}{3}}}{5} + 6\times 8^{\frac{2}{3}} - \frac{3}{8^{\frac{1}{3}}}\right) - \left(\frac{12\times\left(\frac{1}{8}\right)^{\frac{5}{3}}}{5} + 6\times\left(\frac{1}{8}\right)^{\frac{2}{3}} - \frac{3}{\left(\frac{1}{8}\right)^{\frac{1}{3}}}\right)\right] = \frac{461}{2} \Rightarrow \theta = \ldots$ OR $\pi\left[\frac{12x^{\frac{5}{3}}}{5} + 6x^{\frac{2}{3}} - \frac{3}{x^{\frac{1}{3}}}\right]_{\frac{1}{8}}^{8} = \pi\left[\left(\frac{12\times 8^{\frac{5}{3}}}{5} + 6\times 8^{\frac{2}{3}} - \frac{3}{8^{\frac{1}{3}}}\right) - \left(\frac{12\times\left(\frac{1}{8}\right)^{\frac{5}{3}}}{5} + 6\times\left(\frac{1}{8}\right)^{\frac{2}{3}} - \frac{3}{\left(\frac{1}{8}\right)^{\frac{1}{3}}}\right)\right] = ..$ followed by $\frac{\theta}{2\pi}\times\ldots = \frac{461}{2} \Rightarrow \theta = \ldots$	M1	3.1a
	$\theta = \frac{40}{9}$ (radians)	A1	1.1b
		(8)	
	(8 marks)		

Student Response A

(8)



9) $\int_{1/8}^{8}(2x^{1/3}+x^{-2/3})\,dx = \left[\frac{6}{4}x^{4/3}+3x^{1/3}\right]_{1/8}^{8}$

$\left[\frac{6}{4}(8)^{4/3}+3(8)^{1/3}\right]-\left[\frac{6}{4}\left(\frac{1}{8}\right)^{4/3}+3\left(\frac{1}{8}\right)^{1/3}\right]$

$[24+6]-\left[\frac{3}{32}+\frac{3}{2}\right]$

$30-\frac{51}{32}=\frac{909}{32}$ $\quad \pi\int(2x^{1/3}+x^{-2/3})dx$

	$2x^{1/3}$	$x^{-2/3}$
$2x^{1/3}$	$4x^{2/3}$	$+2x^{-1/3}$
$x^{-2/3}$	$x^{-1/3}$	x^{0}

$= 4x^{2/3}+3x^{-1/3}+1$

$\pi\int_{1/8}^{8}(4x^{2/3}+3x^{-1/3}+1)\,dx = \left[\frac{12}{5}x^{5/3}+\frac{9}{2}x^{2/3}+x\right]_{1/8}^{8}$

$\frac{12}{5}(8)^{5/3}+\frac{9}{2}(8)^{2/3}+8=\frac{384}{5}+18+8=\frac{514}{5}$

$\frac{12}{5}\left(\frac{1}{8}\right)^{5/3}+\frac{9}{2}\left(\frac{1}{8}\right)^{2/3}+\frac{1}{8}=\frac{3}{40}+\frac{9}{8}+\frac{1}{8}=\frac{53}{40}$

$\pi\left[\frac{514}{5}-\frac{53}{40}\right]=\pi\left(\frac{4059}{40}\right)=\frac{4059}{40}\pi$

2/8

Examiner Comment: M0M0A0M1A1A0M0A0

There is no correct overall strategy in this response as the volume of revolution that is attempted is never scaled in an attempt to find the required angle. Consequently the first mark and last two marks are lost. It appears there are two attempts, one with y integrated and one with y^2 integrated. the second one (with y^2) is taken arrives at an answer in terms of π so is more complete and is the answer that is scored. In this case the second M is lost as there is no term in $x^{-4/3}$, but there is a correct attempt at integration with at least two terms with fractional indices, with two terms correct following through their expansion, so the method and follow through accuracy are gained.

Student Response B

$V = \pi \int_a^b y^2 \, dx$

$V = \pi \int_a^b (2x^{1/3} + x^{-2/3})^2 \, dx$

$= \pi \int_a^b (4x^{2/3} + x^{-4/3}) \, dx$

$= \pi \left[\frac{4x^{5/3}}{5/3} + \frac{x^{-1/3}}{-1/3} \right]_{1/8}^{8}$

$= \pi \left[\frac{12}{5} x^{5/3} - 3x^{-1/3} \right]_{1/8}^{8}$

$\pi \left[\left[\frac{12}{5}(8)^{5/3} - 3(8)^{-1/3} \right] - \left[\frac{12}{5}\left(\frac{1}{8}\right)^{5/3} - 3\left(\frac{1}{8}\right)^{-1/3} \right] \right] = \frac{3249}{40}\pi$

$\frac{3249\pi}{40} = \frac{461x}{2}$

$3249\pi = 9220x$

$\frac{3249\pi}{9220} = x \qquad x = 1.107053637$

$2\pi \div x = 5.675592489$

$x = 1.107053637$

so angle $\theta = \frac{2\pi}{x}$

or... 5.675592489

5.68 (2 d.p)

4/8

Examiner Comment: M1M0A0M1A1A0M1A0

A correct overall strategy has been applied of using the volume of revolution formula to find the volume of the shape rotated through 2π followed by an attempt to scale the result to find an angle, so the first method is scored.

The expansion of y^2 has only two terms, and so the second M and following A mark are lost, as at least a three-term expression was required. But the two terms are integrated correctly gaining the method and follow through accuracy mark for integration, but the third A mark is lost as there is a term missing. The final M mark is earned as the method to find the required angle is correct, albeit in a complicated way. If the volume had been calculated correctly, the correct answer would have been obtained, but the final mark is lost due to the errors in finding the volume.

Student Response C

$y = 2x^{\frac{1}{3}} + x^{-\frac{2}{3}}$

$y^2 = 4x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} + 2x^{-\frac{1}{3}} + x^{-\frac{4}{3}}$

$\int_{\frac{1}{8}}^{8} 4x^{\frac{2}{3}} + 4x^{-\frac{1}{3}} + x^{-\frac{4}{3}}\,dx$

$\left[\frac{12}{5}x^{\frac{5}{3}} + 6x^{\frac{2}{3}} - 3x^{-\frac{1}{3}}\right]_{\frac{1}{8}}^{8}$

~~$\frac{474}{5}$~~ ~~$\frac{3}{40}$~~

$\frac{993}{10} + \frac{177}{40} = \frac{4149}{40}$

$\frac{4149}{40} \times \frac{\pi}{n} \qquad \frac{4149\pi}{40n} = \frac{461}{2}$

$8298\pi = 18440n$

$\frac{8298\pi}{18440} = \frac{9\pi}{20} \qquad n = \frac{9}{20}$

$360 \times \frac{9}{20} = 162$

$\theta = 162°$

The solid is rotated through 162°

6/8

Examiner Comment: M1M1A1M1A1A1M0A0

This student applied a correct overall strategy of finding the integral of y^2 and attempting to scale at the end, so the first mark was awarded. The expansion and integration is all fully correct, gaining the next 5 marks. However, the scaling used at the end is incorrect losing the final two marks. The correct calculation at the end should be $360/(\frac{9\pi}{20})$ to find the angle in degrees.

Exemplar Question 10

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10. The population of chimpanzees in a particular country consists of juveniles and adults. Juvenile chimpanzees do not reproduce.

In a study, the numbers of juvenile and adult chimpanzees were estimated at the start of each year. A model for the population satisfies the matrix system

$$\begin{pmatrix} J_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix} \qquad n = 0, 1, 2, \ldots$$

where a is a constant, and J_n and A_n, are the respective numbers of juvenile and adult chimpanzees n years after the start of the study.

(*a*) Interpret the meaning of the constant a in the context of the model.

(1)

At the start of the study, the total number of chimpanzees in the country was estimated to be 64 000

According to the model, after one year the number of juvenile chimpanzees is 15 360 and the number of adult chimpanzees is 43 008

(*a*) (i) Find, in terms of a

$$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}$$

(3)

(ii) Hence, or otherwise, find the value of a.

(3)

(iii) Calculate the change in the number of juvenile chimpanzees in the first year of the study, according to this model.

(2)

Given that the number of juvenile chimpanzees is known to be in decline in the country,

(c) comment on the short-term suitability of this model.

(1)

A study of the population revealed that adult chimpanzees stop reproducing at the age of 40 years.

(d) Refine the matrix system for the model to reflect this information, giving a reason for your answer.

(*There is no need to estimate any unknown values for the refined model, but any known values should be made clear*.)

(2)

(Total for Question 10 is 12 marks)

Mean Score 3.8 out of 12

Examiner Comment

This question tested matrix work (3.1,3.5, 3.6) in a modelling context, with an element of problem solving. It was found challenging by many students.

In part (a), a correct interpretation of the constant *a* was rarely seen with many candidates incorrectly believing it to be the number of surviving juveniles, rather than the number of juveniles who remained as juveniles in the following year.

Part (b)(i) was answered well by the majority of candidates, with almost all able to find the determinant of the 2×2 matrix and most going on correctly to find its inverse. A common error was that only one sign change had been made, while some did not form the adjoint matrix at all, but in general 3 was the most common mark here.

Attempts at part (b)(ii) and onwards were less common, but many did nevertheless make some kind of attempt. Of those who did proceed, the majority attempted to use the inverse matrix to determine the values of a and J_0 but had found it a challenge to match up the results of their matrix multiplication with the total of 64,000 chimpanzees. Algebra here, in finding a, often resulted in error. Similar levels of success were achieved by those candidates who set up and solved associated simultaneous equations, and these more commonly achieved the values of J_0 correctly.

For part (b)(iii) many candidates failed to see the connection between this part and (b)(ii) and did not find the value of J_0 unless they had already done so as a part of their method. Even in such cases they did not always go on to achieve the correct answer. But there were also many students who did make good attempts at this part, achieving the correct difference.

In part (c) candidates who achieved a value for J_0 would generally go on to make a comment for this part, although not all were able to make the link correctly. There were a large number of responses that were not based on the solutions from part (b), but instead tried to make general comment about long term unsuitability of the model. Many such of these had no answer to part (b) and so could not access the mark in any case.

Very few students even attempted part (d). Amongst those who did, only a very small minority realised the need to extend from a 2×2 matrix system to a 3×3 system by introducing a third category of chimpanzee, and fully correct answers were very rare. Many focussed more on trying to adapt one entry in the given system, either just in its value, or by adding a variable based on the number of mature chimpanzees, while others attempted to adjust by subtraction of an extra vector term. However, none of these methods would enable the new number of mature chimpanzees to be determined and so no credit could be given for them.

Mark Scheme

Question	Scheme	Marks	AOs
10 (a)	a represents the proportion of juvenile chimpanzees that (survive and) **remain** juvenile chimpanzees the next year.	B1	3.4
		(1)	
(b)(i)	Determinant = $0.82a - 0.08\times0.15$	M1	1.1b
	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} = \ldots\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$	M1	1.1b
	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} = \dfrac{1}{0.82a-0.012}\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$	A1	1.1b
		(3)	
(ii)	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}\begin{pmatrix} 15360 \\ 43008 \end{pmatrix} = \dfrac{1}{0.82a-0.012}\begin{pmatrix} 0.82\times15360-0.15\times43008 \\ (-0.08)\times15360+43008a \end{pmatrix}$ OR forms equations $15360 = aJ_0 + 0.15\times A_0$ $43008 = 0.08\times J_0 + 0.82\times A_0$	M1	3.1a
	$\dfrac{1}{0.82a-0.012}\left[6144+\left(43008a-1228.8\right)\right]=64000$ $\Rightarrow 4915.2+43008a = 64000(0.82a-0.012) \Rightarrow a = \ldots$ OR $A_0 = 64000 - J_0 \Rightarrow 43008 = 0.08\times J_0 + 0.82\times\left(64000-J_0\right) = J_0 = \ldots$ $\Rightarrow a = \dfrac{15360-\left(64000-J_0\right)}{J_0} = \ldots$	M1	3.1a
	$a = \dfrac{5683.2}{9472} = 0.60$	A1	1.1b
		(3)	
(iii)	Initial juvenile population = $\dfrac{\text{"6144"}}{\text{"0.48"}} = 12800$	M1	3.4
	So change of 2560 juvenile chimpanzees	A1	1.1b
		(2)	
(c)	As the number of juveniles has increased, the model is not initially predicting a decline, so is not suitable in the short term. (Follow through their answer to (b) – but they must have made an attempt at it to find at least a value for J_0)	B1ft	3.5a
		(1)	
(d)	Third category needs to be introduced for chimpanzees aged 40 and above, mature chimpanzees M_n, and a matrix multiplication of increased dimension set up. Accept $3\times3, 3\times2$ or 2×3 matrices including all three categories in the column vector.	M1	3.5c

	The corresponding matrix model will have the form $$\begin{pmatrix} J_{n+1} \\ A_{n+1} \\ M_{n+1} \end{pmatrix} = \begin{pmatrix} a & b & \underline{\mathbf{0}} \\ 0.08 & c & 0 \\ 0 & d & e \end{pmatrix} \begin{pmatrix} J_n \\ A_n \\ M_n \end{pmatrix}$$ (The underlined zero must be correct but do not be concerned about any values used in the other entries.)	A1	3.3
		(2)	
			(12 marks)

Notes

(a)	**B1**	Correct interpretation. Need not mention survival but must be clear it is the (proportion of) **juveniles that remain as juveniles** the next year (ie those that survive but don't progress to adulthood). E.g. accept "(number of) juveniles who do not become adults" but do not accept "surviving juveniles".
		Mark part (b) as a whole.
(b)(i)	**M1**	Attempts the determinant in terms of a Allow miscopies for the attempt. Allow $0.82a - 0.12$ as a slip.
	M1	Attempts the form of the inverse, swapped leading diagonals and sign changed on both off diagonals. Allow miscopies of the numbers but the signs must be correct.
	A1	Correct inverse matrix
(ii)	**M1**	Use the inverse matrix and attempts to find the initial juvenile and adult populations. . (May have determinant 1 for this mark.) Alternatively, sets up simultaneous equations from the original system, $15360 = aJ_0 + 0.15 \times A_0$ and $43008 = 0.08 \times J_0 + 0.82 \times A_0$ Accept with J_n and A_n or other appropriate variables.
	M1	Uses the sum of initial populations equals 64000 in an attempt to find a. (May have determinant 1 for this mark.) If using alternative, use of e.g. $A_0 = 64000 - J_0$ in second equation to find J_0, followed by attempt to find a. Award for an attempt to solve the equations, but don't be too concerned with the algebraic process as long as they are attempting to use all three equations.
	A1	Correct value, $a = 0.6$ (or 0.60 or $\frac{3}{5}$).
(iii)	**M1**	Uses their a to find the value of J_0. This mark may be gained for work done in (ii) if the alternative has been used but must have come from a correct method.
	A1	Correct difference found, as long as there is no contradictory statement – so "decrease of 2560" is A0.
(c)	**B1ft**	Comments that the change is an increase so does not fit the model. Follow through their answer to (b) as long as at least a value for J_0 has been found. If a decrease has been found allow for commenting the model is suitable. If an answer is given to (b)(iii), follow through on whatever their answer is. If no answer has been given, but an initial population found, a comparison should be made between this value and 153600 with conclusion must be consistent with their answer for J_0
(d)	**M1**	Introduces a third category (may be *M*ature, *E*lderly or any suitable letter used) and sets up a matrix multiplication (the left hand side may be missing for this mark) with all three categories in the column vector. The dimension of the matrix should be 3 in at least either row or column, and there should be a 3×1 vector.
	A1	Sets up the new matrix equation, including both sides and making clear the zero (underlined) so that the correct progression that no new juveniles arise from the mature chimpanzees is clear. Overlook other values, though ideally the other two zeroes are shown too, to indicate mature chimpanzees do not regress to adulthood, and juveniles cannot proceed directly to mature chimpanzees.

Student Response A

a) a is the surviving population of chimpanzees (that dont become adults)

b) i) $J_{n+1} = aJ_n + 0.15A_n$

↑ surviving population ↑ adults that give birth

b) $\det(M) = 0.82a - 0.12$

$$= \frac{1}{0.82a - 0.12}\begin{pmatrix} 0.82 & -0.15 \\ -0.05 & a \end{pmatrix}$$

ii

~~15360~~ $A_{n+1} = 0.08J_n + 0.82A_n$

↑ amount of juveniles that become adults ↑ surviving Adults

$43008 = 0.08J_n + 0.82A_n \rightarrow A_n$

$15360 = aJ_n + 0.15A_n$

$43008 = 0.08(64000) - 0.08A_n + 0.82A_n$

$J_n + A_n = 64000$

$J_n = 64000 - A_n$

~~43008 = 0.08a~~

$64000 = 1J_n + 1A_n$

$58368 = aJ_n + 0.15A_n + 0.08J_n + 0.82A_n$

$58368 = (a + 0.08)J_n + (0.97)A_n$

$64000 - J_n = A_n$

$58368 = (a + 0.08)J_n + 0.97(64000) - 0.97J_n$

$58368 - 62080 = (a + 0.08 - 0.97)J_n$

$-3715 = -0.89J_n$

$-3714.11 = aJ_n$

$\frac{3714.11}{64000}$ ~~x100~~ $\vdash a \mp 0.058$ ~~a = 0.942~~

$a = 0.92$

c) In the short term, this model is valid as it predicts a decline in the overall population

d) $\begin{pmatrix} J_{n+1} \\ A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0.15 - B_n \\ 0.08 & 0.82 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix}$ where B_n is the amount of chimpanzees that are 40 years or older

~~where B_n is the amount of chimpanzees over 40~~

4/12

Examiner Comment: (a) B1 (b)(i)M1M1A0 (ii) M1M0A0 (iii) M0A0 (c) B0 (d) M0A0

An acceptable interpretation is given for part (a) referring to *a* being about the (surviving juvenile) chimpanzees that don't become adults. This was a marginal decision as there is no direct reference to juveniles, and no mention of proportion, but the parenthetical comment was that it was about those not become adults that was deemed the key point for the mark. That surviving juveniles was meant was taken as being implied.

An attempt at the determinant is made in (b)(i), allowing the slip in the 0.012 for the method mark, and the adjoint matrix is correctly attempted, the copying error in the bottom left entry again affecting only the accuracy rather than the method. So both method marks were awarded in this case, though the inverse is incorrect, so the accuracy is lost.

In (b)(ii) the alternative approach via simultaneous equations is attempted, and the required equations are set up for the first method mark. However, the next mark was not give as there is no correct method that proceeds to find either J_0 or A_0 first before then using this to find *a*.

There is no attempt to find a difference in populations so no marks can be scored in (b)(iii) and the mark for (c) requires a value for J_0 to have been found, so this is also not accessible for this response.

The mark in (d) cannot be awarded as, although a third category of over 40 chimpanzees is introduced, the matrix systems is not increased in dimension in either direction to allow for interaction of the new category.

Student Response B

a. a is the number of juvenile chimpanzees that ~~transition~~ remain into an ~~adult~~ adult every year.

b. i. $(a \times 0.82) - (0.15 \times 0.08) = 0.82a - 0.012$

$$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} = \begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$$

$$= \frac{1}{0.82a - 0.012}\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$$

ii. $$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}\begin{pmatrix} 15360 \\ 43008 \end{pmatrix} = \begin{pmatrix} \frac{0.82}{0.82a - 0.012}(15360) + \frac{(43008)(0.15)}{0.82a - 0.012} \\ \frac{-0.08(15360)}{0.82a - 0.012} + 43008a \end{pmatrix}$$

$$\frac{1}{0.82a - 0.012}\left(12595.2 - 6451.2 - 1228.8 + 43008a\right)$$

$$= 64000$$

iii. $aJ + 0.15A = 15360$

$0.08J + 0.82A = 43008$

$A + J = 64000 \qquad A = 6400 - J$

$aJ + 6400 - J = 15360$

$aJ - J = 15360 - 6400 = 8960$

$J(a-1) = 8960$

$a = \frac{8960}{J} + 1$

$\Rightarrow J = 12800$

$A = 51200$

$a = \frac{8960}{12800} + 1 = \frac{17}{10} = 1.7$

$$\begin{pmatrix} 1.7 & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} \begin{pmatrix} 15360 \\ 43008 \end{pmatrix}$$

$$= \begin{matrix} 4495.730 \\ 52015.05 \end{matrix}$$

c. It's not sustainable as it shows the number of juvenile monkeys to be increasing not ~~decreasin~~ decreasing

d. $\begin{pmatrix} a & 0.15 \times 0.4 \\ 0.08 & 0.82 \end{pmatrix}^{-} - \begin{pmatrix} a & 0.08 \\ 0.08 & 0.82 \end{pmatrix}$ ~~0.06~~

6/12

Examiner Comment: (a) B0 (b)(i)M1M1A1 (ii) M1M1A0 (iii) M1A0 (c) B0 (d) M1A0

The mark in (a) is not earned as an incorrect interpretation is given in part (a) that it is the number of chimpanzees becoming adult. Note that number of chimpanzees remaining juvenile would have been accepted for the answer even though it is the proportion rather than number.

A correct inverse matrix is given in part (b)(i) for all three marks therein.

For (b)(ii) there are two attempts and the second one is scored as it is the one that is most complete since it proceeds to a value for *a*, whereas the first attempt does not reach a value. This solution proceeds via the alternative approach of forming simultaneous equations for the initial juvenile and adult chimpanzee populations from the given information (scoring the first M) and then combing with the fact the sum of initial populations is 64000 and attempting to use this to find *a*. In this case J_0 is found first and then used in an equation to solve for *a*. This gains the second M mark, but the value of *a* is incorrect, so the accuracy is lost.

The method in part (b)(iii) is scored from the work in (ii) in finding the correct value of J_0 (from solving the two equations in *J* and *A*). Alternative it could be scored for using their value of *a* in the inverse matrix and multiplying by the populations from year one to find the initial populations. There is no attempt to find the change in juvenile populations, however, so the accuracy mark is lost.

The mark in part (c) is not awarded since the working of the student implies the initial population of juveniles is 4445.730, which would represent a decrease and not an increase as claimed, so the conclusion is inconsistent with the working. Although the correct value of J_0 was found earlier, it was never identified as the initial population of juveniles.

No marks are scored in part (d) as there is no introduction of a third category of chimpanzees.

Student Response C

a) The number of juveniles born each year.

b) i) n=0, 64000

$$\begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} 15360 \\ 43009 \end{pmatrix} = \begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \end{pmatrix}$$

~~det A:~~ det = 0.82a − 0.012

$$\frac{1}{0.82a-0.012}\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$$

ii) $M^{-1} \times \begin{pmatrix} J_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} J_n \\ A_n \end{pmatrix}$

$$\frac{1}{0.82a-0.012}\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}\begin{pmatrix} 15360 \\ 43008 \end{pmatrix} = \begin{pmatrix} J_0 \\ A_0 \end{pmatrix}$$

$\frac{6144}{0.82a-0.012} = J_0$ $\qquad$ $J_0 + A_0 = 64000$

$\frac{-1229.8 + 43009a}{0.82a-0.012} = A_0$

$$\frac{6144}{0.92a - 0.012} + \frac{-1228.8 + 43008a}{0.92a - 0.012} = 64000$$

$$6144 - 1228.8 + 43008a = 64000(0.92a - 0.012)$$

$$4915.2 + 43008a = 58880a - 768$$

$$5683.2 = 9472a$$

$$\therefore a = 0.6.$$

(ii) $J_0 = 12800$ 15360

$$\left(\frac{15360}{12800} - 1\right) \times 100 = 20$$

2560 more

20% change

c) Short term accurate as the impact of trips to help the decline until the aspect acted later.

d)
$$\begin{pmatrix} 0.6 & 0.15 \\ 0.08 & 0.92 \\ x & 0 \end{pmatrix} \begin{pmatrix} J_n \\ A_n \\ S_n \end{pmatrix}$$

S_n = Senior chimpanzees

$0 \Rightarrow$ not reproducing

$x \Rightarrow$ number dying each year

9/12

Examiner Comment: (a) B1 (b)(i)M1M1A1 (ii) M1M1A1 (iii) M1A1 (c) B0 (d) M1A1

An incorrect interpretation of *a* is given in part (a), no reference to juveniles remaining juvenile is made, so the first mark is lost.

Part (b) is fully correct, with determinant and inverse matrix correctly identified and main method of the scheme, using the inverse matrix to find the pre-image of the second year data and summing the two sub-populations to the required total to find *a* in part (ii). The correct value of J_0 is identified (implying the method mark) with a change of 2560 given (and no contradictory comments about it).

Part (c) does not score the mark as the answer given makes no reference to what the model has predicted or the suitability thereof as a result.

For part (d) the student has shown a realisation of the need to introduce a third category of senior chimpanzees and attempted to expand the matrix dimension in at least one direction, so the method mark is awarded, but the matrix system created is not correct (and does not have the left hand side and zeroes required), so the final accuracy mark is lost.

A Level Further Mathematics – Core Pure 1 (9FM0 01)

Exemplar Question 1

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1.

$$f(z) = z^4 + az^3 + bz^2 + cz + d$$

where a, b, c and d are real constants.

Given that $-1 + 2i$ and $3 - i$ are two roots of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram,

(4)

(b) find the values of a, b, c and d.

(5)

(Total for Question 1 is 9 marks)

Mean Score 8.3 out of 10

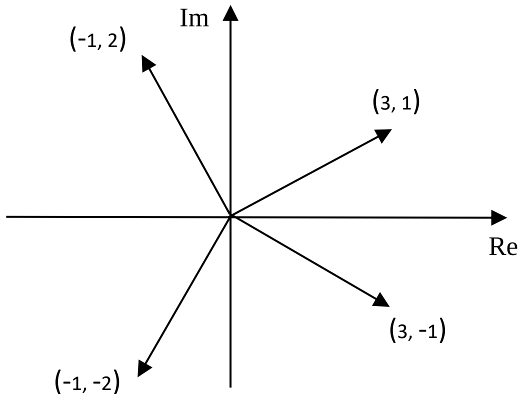
Examiner Comments

This question is assessing complex numbers (spec ref 2.3, 2.4) - non-real roots of a polynomial occur as conjugate pairs and representing complex numbers on an Argand diagram.

Part (a) was accessible to almost all candidates, with most gaining full marks. A handful of candidates thought the conjugate was 1 + 2i and – 3 – i and a few candidates lost marks because of a poor diagram, failing to indicate a scale or labelling or with -1 + 2i closer to the real axis than 3 + i. Advice to candidates is to make sure that the end coordinate is clearly labelled.

Part (b) proved to be more challenging for some candidates. The most common approach was attempt to find the sum, pair sum, triple sum and product of the four roots. Most were able to find a correct sum and product, but errors were sometimes made when attempting to find the pair and triple sums. Some errors were due to a missing pair or triple sum, but most errors occurred in the algebraic manipulation of the terms. Most candidates realised they needed a = -sum, but a few candidates omitted the minus on the triple sum, and so lost the last M mark.
The intention of this question was for candidates to use the conjugate roots to find two quadratics using x^2 – (sum of roots)x + (product of roots) and then multiply to find the equation of the quartic.
A significant number of candidates attempted this method and most were able to form the correct quadratics and then go on to multiplying these out to obtain a quartic. Common errors with this approach were to make a sign error when attempting to apply $i^2 = -1$, or to apply "+ sum" rather than "- sum" for the x-term in the quadratic. Some candidates made slips when multiplying out the quadratic factors, and one or two lost the final A mark for stating a quartic in term of x and not z.
A few chose to substitute roots into a general quartic obtained two or more simultaneous equations, but often these contained errors. Most then failed to correctly solve their equations to find a, b, c and d.

Mark Scheme

Question	Scheme	Marks	AOs
1(a)	$z=-1-2\mathrm{i}$ or $z=3+\mathrm{i}$	M1	1.2
	$z=-1-2\mathrm{i}$ and $z=3+\mathrm{i}$	A1	1.1b
		B1 B1	1.1b 1.1b
		(4)	
(b) **Way 1**	$(z-(-1+2\mathrm{i}))(z-(-1-2\mathrm{i}))=$ or $(z-(3+\mathrm{i}))(z-(3-\mathrm{i}))=\ldots$ $\mathrm{f}(z)=(z-(-1+2\mathrm{i}))(z-(-1-2\mathrm{i}))(z-(3+\mathrm{i}))(z-(3-\mathrm{i}))=\ldots$	M1	3.1a
	z^2+2z+5 or $z^2-6z+10$ e.g. $\mathrm{f}(z)=(z^2+2z+5)(\ldots)$	A1	1.1b
	z^2+2z+5 and $z^2-6z+10$ $\mathrm{f}(z)=(z^3+z^2(-1-\mathrm{i})+z(-1+2\mathrm{i})-15-5\mathrm{i})(\ldots)$	A1	1.1b
	$\mathrm{f}(z)=(z^2+2z+5)(z^2-6z+10)$ Expands the brackets to forms a quartic	M1	3.1a
	$\mathrm{f}(z)=z^4-4z^3+3z^2-10z+50$ or States $a=-4, b=3, c=-10, d=50$	A1	1.1b
		(5)	
Way 2	sum roots $=\alpha+\beta+\gamma+\delta=(-1+2\mathrm{i})+(-1-2\mathrm{i})+(3+\mathrm{i})+(3-\mathrm{i})=\ldots$ pair sum $=\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$ $=(-1+2\mathrm{i})(-1-2\mathrm{i})+(-1+2\mathrm{i})(3-\mathrm{i})+(-1+2\mathrm{i})(3+\mathrm{i})+(-1-2\mathrm{i})(3-\mathrm{i})$ $+(-1-2\mathrm{i})(3+\mathrm{i})+(3+\mathrm{i})(3-\mathrm{i})=\ldots$ triple sum $=\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\alpha\gamma\delta$ $=(-1+2\mathrm{i})(-1-2\mathrm{i})(3-\mathrm{i})+(-1+2\mathrm{i})(-1-2\mathrm{i})(3+\mathrm{i})+(-1+2\mathrm{i})(3+\mathrm{i})(3-\mathrm{i})$ $+(-1-2\mathrm{i})(3+\mathrm{i})(3-\mathrm{i})=\ldots$ Product $=\alpha\beta\gamma\delta=(-1+2\mathrm{i})(-1-2\mathrm{i})(3-\mathrm{i})(3+\mathrm{i})=\ldots$	M1	3.1a
	sum = 4, pair sum = 3, triple sum = 10 and product = 50	A1 A1	1.1b 1.1b

	$a=-(\text{their sum roots})=-4$ $b=+(\text{their pair sum})=3$ $c=-(\text{triple sum})=-10$ $d=+(\text{product})=50$	M1 A1	3.1a 1.1b
		(5)	
Way 3	$f(z)=(-1+2i)^4+a(-1+2i)^3+b(-1+2i)^2+c(-1+2i)+d=0$ $f(z)=(3+i)^4+a(3+i)^3+b(3+i)^2+c(3+i)+d=0$	M1	3.1a
	Leading to $-7+11a-3b-c+d=0$ $\quad 24-2a-4b+2c=0$ $28+18a+8b+3c+d=0$ $\quad 96+26a+6b+c=0$	A1 A1	1.1b 1.1b
	Solves their simultaneous equation to find a value for one of the constants	M1	3.1a
	$a=-4, b=3, c=-10, d=50$	A1	1.1b
		(5)	
			(9 marks)

Notes

(a)

M1: Identifies at least one correct complex conjugate as another root (can be seen/implied by Argand diagram)

A1: Both complex conjugate roots identified correctly (can be seen/implied by Argand diagram)

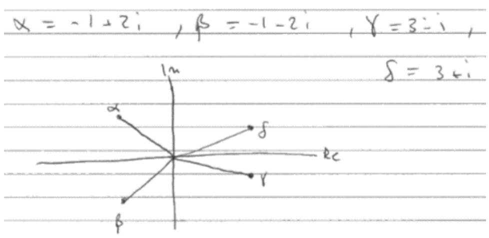
For the next two marks allow either a cross, dot or line drawn where the end point is labelled with the correct coordinate, corresponding complex number or clearly plotted with correct numbers labelled on the axis or indication of the correct coordinates by use of scale markers. Condone (3, i) etc. The axes do not need to be labelled with Re and Im.

B1: One complex conjugate pair correctly plotted.

B1: Both complex conjugate pair correctly plotted. The $3\pm i$ must be closer to the real axes than the $-1\pm 2i$

If there is no indication of the coordinates, scale or complex numbers on the Argand diagram this is B0 B0.

Do accept correct labelling e.g.



(b)

Way 1

M1: Correct strategy for forming at least one of the quadratic factors. Follow through their roots.

A1: At least one correct simplified quadratic factor.

A1: Both simplified quadratic factors correct or a correct simplified cubic factor

M1: A complete strategy to find values for *a*, *b*, *c* and *d* e.g. uses their quadratic factors or cubic and linear factor to form a quartic.
A1: Correct quartic in terms of *z* or correct values for *a*, *b*, *c* and *d* stated.

Way 2
M1: Correct strategy for finding at least three of the sum roots, pair sum, triple sum and product. Follow through their roots. This can be implied by at least three correct values for the sum roots, pair sum, triple sum and product with no working shown. If the calculations are not shown for the sums and product and they have at least two incorrect values this is M0.
A1: At least two correct values for the sum roots, pair sum, triple sum or product.
A1: All correct values for the sum, pair sum, triple sum and product.
M1: Must have real values of *a*, *b*, *c* and *d* and use $a = -$their sum roots, $b =$ their pair sum, $c = -$their triple sum and $d =$ their product.
A1: Correct quartic in terms of *z* or correct values for *a*, *b*, *c* and *d* stated.

Way 3
M1: Substitutes two roots into $\mathrm{f}(z) = 0$ and equates coefficients to form 4 equations
A1: At least two correct equations.
A1: All four correct equations
M1: Solve their four equation (using calculator) to find at least one value. This will need checking if incorrect equations used.
A1: Correct quartic in terms of *z* or correct values for *a*, *b*, *c* and *d* stated.

Note: Correct answer only will score 5/5

Student Response A

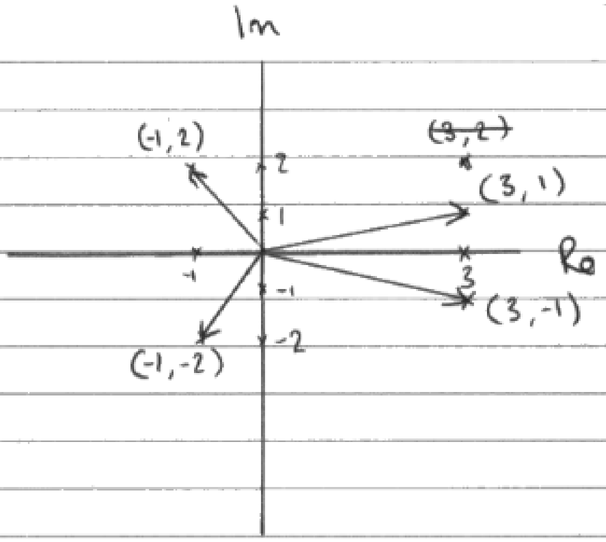
~~$-1+2i$ ... $-1-2i$~~

$z_1 = -1+2i$

$z_1^* = -1-2i$

$z_2 = 3-i$

$z_2^* = 3+i$



~~$\alpha+\beta+\gamma+\mu=$~~

~~$\alpha\beta\gamma\mu = \frac{b}{a}$~~

~~$\Sigma\ \alpha\beta+\alpha\gamma+\beta\mu+\beta\gamma+\beta\mu+\gamma\mu=$~~

~~$\Sigma\alpha$~~ $\Sigma\alpha = -\frac{b}{a}$ ~~[illegible]~~

$\Sigma\alpha\beta = \frac{c}{a}$

$\Sigma\alpha\beta\gamma = -\frac{d}{a}$

$\alpha\beta\gamma\mu = \frac{e}{a}$

$\Sigma\alpha = -\frac{b}{a} = (-1+2i)+(-1-2i)+(3-i)+(3+i)$

$= 4$

$-b = 4$

$b = -4$

4/9

Examiner Comments

In part (a)
M1 A1: Correct complex conjugates stated.
B1 B1: All the roots plotted with correct end coordinates.

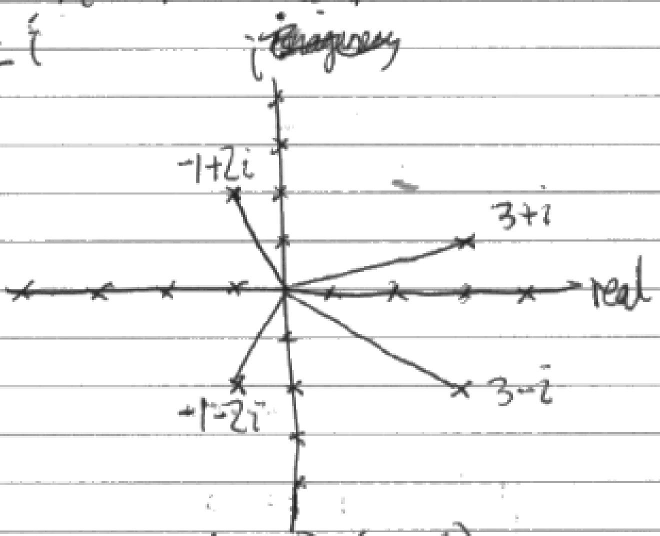
In part (b)
M0 A0 A0: To score the first method mark at least three of the sum, pair sum, triple sum and product need to be found using a correct method. Only one value is found here.
M0 A0: They do have b = - sum of roots but only one value found.

Student Response B

~~$(-1+2i)(3-i) = -3 + i + 6i - 2i^2$~~

~~$= 7i - 1$~~

$-1 \pm 2i \quad 3 \pm i$



~~$(-1+2i)(-1-2i)$~~ ~~$(3-i)(3+i)$~~

~~$= 1 - 4i^2 = 5$~~ $= 5$ ~~$9 + 1 = 10$~~ $= 10$

$(x - (-1+2i))(x - (-1-2i)) = (x + 1 - 2i)(x + 1 + 2i)$

$= x^2 + (1+2i)x + (1-2i)$ ~~$x + 5$~~

$+ (1-2i)(1+2i) =$ ~~5~~

$x^2 + (1+2i)x + (1-2i)x + 5$

$(x - (3+i))(x - (3-i))$

$= x^2 - (3+i)x - (3-i)x + 10$

$-3 - 3i - 3 + 3i = 0$

$1 + 2i + 1 - 2i = 1$

$(x^2 + 10)(x^2 + x + 5)$

$= x^4 + x^3 + 5x^2 + 10x^2 + 10x + 50$

$= x^4 + x^3 + 15x^2 + 10x + 50$

$a = 1, \; b = 15, \; c = 10, \; d = 50$

5/9

Examiner Comments

In part (a)

M1 A1: Correct complex conjugates stated.

B1: The roots –1 + 2i and –1 – 2i are plotted correctly.

B0: The root 3 – i is incorrectly plotted (it must be closer to the x-axis that –1 – 2i)

In part (b) Mark scheme way 1

M1: Attempting to find the quadratic equation for each conjugate pair (x – root) (x – conjugate root)

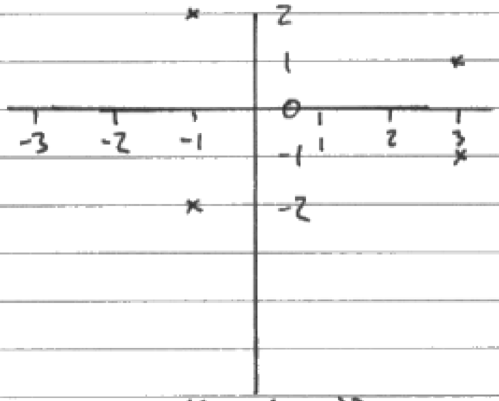
A0 A0: Incorrect quadratics

M1: Multiplies out their quadratics

A0: Incorrect final answer

Student Response C

a) ~~$(x-(-1+2i))(x$~~ $-1+2i$, $-1-2i$, $3-i$, $3+i$



b) $(x-(-1+2i))(x-(-1-2i)) = (x+1-2i)(x+1+2i)$

$= x^2 + x + 2ix$
$\quad + x \quad + 1 \quad + 2i$
$\quad - 2ix \quad + 4 \quad - 2i$

$= x^2 + 2x + 5$

$(x-(3-i))(x-(3+i)) = (x-3+i)(x-3-i) =$

$= x^2 - 3x - ix$
$\quad - 3x \quad + 9 \quad + 3i$
$\quad + ix \quad + 1 \quad - 3i$

$= x^2 - 6x + 10$

$(x^2+2x+5)(x^2-6x+10) = x^4 - 6x^3 + 10x^2$
$\quad + 2x^3 - 12x^2 + 20x$
$\quad 5x^2 - 30x + 50$

$= x^4 - 4x^3 + 3x^2 - 10x + 50$

$a = -4 \quad b = 3 \quad c = -10 \quad d = 50$

9/9

Examiner Comments

In part (a)
M1 A1: Correct complex conjugates stated.
B1 B1: All the roots plotted with correct coordinates.

In part (b) Mark scheme way 1
M1: Attempting to find the quadratic equation for each conjugate pair (x – root) (x – conjugate root)
A1 A1: Both quadratics are correct
M1: Multiplies out their quadratics
A1: All values are correct.

Exemplar Question 2

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2. Show that

$$\int_0^{\infty} \frac{8x - 12}{(2x^2 + 3)(x + 1)} \mathrm{d}x = \ln k$$

where k is a rational number to be found.

(7)

(Total for Question 2 is 7 marks)

Mean Score 3.2 out of 7

Examiner Comments

This question is assessing integration using partial fractions (spec ref 5.4), note that the guidance 'extends to quadratic factors of ax^2 + c in the denominator'

Most candidates realised that they needed to use partial fractions to attempt this question. Those that did not usually scored 0/7.

The question was answered quite well by candidates who chose the correct form of the partial fractions $\frac{Ax+B}{2x^2+3} + \frac{C}{x+1}$ usually scored the first three marks. Most were then able to integrate correctly, though some had the wrong coefficient for the $\ln(2x^2 + 3)$ term.

Many candidates chose the incorrect partial fraction either $\frac{A}{2x^2+3} + \frac{B}{x+1}$ or $Ax(2x^2 + 3) + \frac{B}{x+1}$ and consequently lost the first four marks.

Candidates who chose $\frac{Ax}{2x^2+3} + \frac{B}{x+1}$ sometimes managed to continue to score the M1B1 for combining log terms and the correct upper limit.

Incorrect partial fractions often led to an incorrect arctan integral.

Many candidates missed the next key stage of working for applying the log rules- either before substituting or by substituting the variable 't' then collecting their log functions in terms of 't' before an attempt to simplify. Some of those candidates who did manage to combine their log terms then failed to deal with the limit correctly, believing that this simplified to ln(1). A small number of candidates who had successfully dealt with the limit left their final answer as 2ln(2/3) rather than putting this in the required form.

Many candidates failed to obtain the B1 mark (and hence the final A1 mark) by not recognising the dominant terms. A very common incorrect answer was ln(1/9)

There were some, concise, completely correct attempts at this question.

Mark Scheme

Question	Scheme	Marks	AOs
2	$\frac{8x-12}{(2x^2+3)(x+1)} = \frac{Ax+B}{2x^2+3} + \frac{C}{x+1}$	M1	3.1a
	$8x-12 = (Ax+B)(x+1) + C(2x^2+3)$ E.g. $x=-1 \Rightarrow C=-4, x=0 \Rightarrow B=0, x=1 \Rightarrow A=8$ Or Compares coefficients and solves $(A+2C=0 \quad A+B=8 \quad B+3C=-12)$ $\Rightarrow A=..., B=..., C=...$	dM1	1.1b
	$A=8 \quad B=0 \quad C=-4$	A1	1.1b
	$\int\left(\frac{8x}{2x^2+3} - \frac{4}{x+1}\right)\mathrm{d}x = 2\ln(2x^2+3) - 4\ln(x+1)$	A1ft	1.1b
	$2\ln(2x^2+3) - 4\ln(x+1) = \ln\left(\frac{(2x^2+3)^2}{(x+1)^4}\right)$ or $2\ln(2x^2+3) - 4\ln(x+1) = 2\ln\left(\frac{(2x^2+3)}{(x+1)^2}\right)$	M1	2.1
	$\lim_{x\to\infty}\left\{\ln\frac{(2x^2+3)^2}{(x+1)^4}\right\} = \ln 4$ or $\lim_{x\to\infty}\left\{2\ln\frac{(2x^2+3)}{(x+1)^2}\right\} = 2\ln 2$	B1	2.2a
	$\Rightarrow \int_0^\infty \frac{8x-12}{(2x^2+3)(x+1)}\mathrm{d}x = \ln\frac{4}{9}$ cao	A1	1.1b
		(7)	
		(7 marks)	

Notes

M1: Selects the correct form for partial fractions.

dM1: Full method for finding values for all three constants. Dependent on having the correct form for the partial fractions. Allow slips as long as the intention is clear.

A1: Correct constants or partial fractions.

A1ft: Integrates $\int \frac{px}{2x^2+3} - \frac{q}{x+1}\,dx = \frac{p}{4}\ln(2x^2+3) - q\ln(x+1)$ and no extra terms

M1: Combines two algebraic log terms correctly.

B1: Correct upper limit for $x \rightarrow \infty$ by recognising the dominant terms. (Simply replacing x with ∞ scores B0). This can be implied.

A1: Deduces the correct value for the improper integral in the correct form, cao A0 for $2\ln\frac{2}{3}$

Correct answer with no working seen is no marks.

Note: Incorrect partial fraction form,

$\frac{A}{2x^2+3} + \frac{B}{x+1}$ or $\frac{Ax}{2x^2+3} + \frac{B}{x+1}$ the maximum it can score is M0M0A0A0M1B1A0

Student Response A

$$\frac{8x-12}{(2x^2+3)(x+1)} = \frac{A}{2x^2+3} + \frac{B}{x+1}$$

$$8x-12 = (x+1)A + (2x^2+3)B$$

let $x = -1$

$$-20 = 0A + 2(-1)^2 + 3B$$

$$-20 = 5B, \quad B = -4$$

Compare x' coefficients

$$8x = Ax, \Rightarrow A = 8$$

$$\Rightarrow \frac{8x-12}{(2x^2+3)(x+1)} = \frac{8}{(2x^2+3)} - \frac{4}{x+1}$$

$$\int_0^\infty \frac{8x-12}{(2x^2+3)(x+1)}\,dx = 4\int_0^\infty \frac{2}{2x^2+3} - \frac{1}{(x+1)}\,dx$$

$$= \lim_{t\to\infty} 4\int_0^t 2(2x^2+3)^{-1} - (x+1)^{-1}\,dx$$

$$X \to \lim_{t\to\infty} 4\left[\ln(2x^2+3) - \ln(x+1)\right]_0^t$$

$$= \lim_{t\to\infty} 4\left[\ln\left(\frac{2x^2+3}{x+1}\right)\right]_0^t$$

$$= \lim_{t\to\infty} 4\left(\ln\left(\frac{2t^2+3}{t+1}\right) - \ln 3\right)$$

$$\lim_{t\to\infty}, \ \ln\left(\frac{2t^2+3}{t+1}\right) \to \ln(1)$$

$$4(\ln 1 - \ln 3)$$

$$= 4\ln(1/3)$$

$$= \ln 1/81$$

$$, \ k = 1/81$$

1/7

Examiner Comments

The incorrect form of the partial fraction scores M0 M0 A0 A0.
M1 for combining two log algebraic log terms
B0 incorrect upper limit
A0 Incorrect answer
This was a typical response.

Student Response B

$$\frac{8x-12}{(2x^2+3)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{2x^2+3}$$

$$8x-12 = A(2x^2+3) + (Bx+C)(x+1)$$

coeff of x^2: $A + B = 0$ ①

coeff of x: $B + C = 8$ ② $\rightarrow C = 8 - B$

constant: $3A + C = -12$ ③

sub ② into ③

$3A + 8 - B = -12$

$3A - B = -20$ ③

Solve ① and ③

① + ③ $4A = -20$ $A = -5$

sub into ③ $-15 + C = -12$

$C = 3$

sub into ② $B + 3 = 8$

$B = 5$

$$\frac{8x-12}{(2x^2+3)(x+1)} = \frac{5x+3}{(2x^2+3)} - \frac{5}{(x+1)}$$

$\lim_{t\to\infty}\int_0^t \frac{5x+3}{(2x^2+3)} - \frac{5}{(x+1)}\,dx$

let $y = \frac{1}{a}\arctan\frac{x}{a}$

$a\tan ya = x$

$dy\, a^2\sec^2(ya) = dx$

$\frac{dy}{dx} = \frac{1}{a^2+a^2\tan^2 y} = \frac{1}{a^2+x^2}$

$\lim_{t\to\infty}\int_0^t \frac{5x}{2x^2+3} + \frac{3}{2x^2+3} - \frac{5}{x+1}\,dx$

$\frac{1}{x^2+\frac{3}{2}}$:

$\lim_{t\to\infty}\left[\frac{5}{4}\ln(2x^2+3) - 5\ln(x+1) + \frac{3\sqrt{2}}{2\sqrt{3}}\arctan\frac{\sqrt{2}x}{\sqrt{3}}\right]_0^t$

$= \lim_{t\to\infty}\left[\ln\frac{(2x^2+3)^{5/4}}{(x+1)^5} + \frac{3\sqrt{2}}{2\sqrt{3}}\arctan\left(\frac{\sqrt{2}}{\sqrt{3}}x\right)\right]_0^t$

$t\to\infty \quad \left(\frac{(2x^2+3)^{5/4}}{(x+1)^5}\right)\to 0 \quad \ln 0 = 1$

$t\to\infty \quad \arctan t \to \frac{\pi}{4}$

$I = \left(1 + \frac{3\sqrt{2}}{2\sqrt{3}}\frac{\pi}{4}\right) - \ln\frac{3^{5/4}}{1} \mp 0$

$= 1 + \frac{3\sqrt{2}\,\pi}{8\sqrt{3}} - \frac{5}{4}\ln 3$

3/7

Examiner Comments

M1 M1: For correct form of the partial fraction and an attempt to find the constants by equating coefficients

A0: Incorrect constants

A0ft: To score the follow thorough integration mark the candidate needed to have the correct form for $\int \frac{px}{2x^2+3} - \frac{q}{x+1}\,dx$

M1: For combining two log algebraic log terms

B0: incorrect upper limit

A0: Incorrect answer

Student Response C

$$\frac{8x-12}{(2x^2+3)(x+1)} = \frac{Ax+B}{2x^2+3} + \frac{C}{x+1}$$

$C(x+1) + (Ax+B)(2x^2+3) = 8x-12$

x^2 $2A=0$ $2A=0$ $2A+2$

x $C+3A$

x^3 terms $2A=0$ $A=0$

x^2 $3A+2B=0$

x $C+3A=8$

cnt. $C+3B=-12$

$C(2x^2+3) + (Ax+B)(x+1) = 8x-12$

x^2: $2C+A=0$ $2C=-A$

x: $A+B=8$

cnt. $3C+B=-12$

$-2C+B=8$ ①

$3C+B=-12$ ②

① − ②

$-2C-3C = 8 - -12$

$-5C = 20$

$C=-4$

$2(-4) = -A$

$A=8$

$8+B=8$

$B=0$

$$\frac{8x-12}{(2x^2+3)(x+1)} = \frac{8x}{2x^2+3} - \frac{4}{x+1}$$

$$I = \int_0^\infty \frac{8x-12}{(2x^2+3)(x+1)}\,dx = \int_0^\infty \frac{8x}{2x^2+3} - \frac{4}{x+1}\,dx$$

$$= \lim_{t\to\infty} \int_0^t \frac{8x}{2x^2+3} - \frac{4}{x+1}$$

$$= \lim_{t\to\infty} \Big[2\ln|2x^2+3| - 4\ln|x+1|\Big]_0^t$$

$$= \lim_{t\to\infty} \left[2\ln\left|\frac{2x^2+3}{(x+1)^2}\right|\right]_0^t$$

$$x^2+2x+1 \;\overline{)\;2x^2+3}\quad 2$$
$$-(2x^2+4x+2)$$
$$-4x+1$$

$$= \lim_{t\to\infty} \left[2\ln\left|2+\frac{1-4x}{x^2+2x+1}\right|\right]_0^t$$

$$= \lim_{t\to\infty} \left[2\ln\left|2+\frac{1}{(x+1)^2} - \frac{4}{x+2+\frac{1}{x}}\right|\right]_0^t$$

$$\lim_{t\to\infty}\left(\frac{1}{(x+1)^2}\right) \to 0$$

$$\lim_{t\to\infty}\left(-\frac{4}{x+2+\frac{1}{x}}\right) \to 0$$

So therefore

$I = 2\ln 2 - 2\ln|2+1|$

$= 2\ln\left|\frac{2}{3}\right|$ -0.81

6/7

Examiner Comments

M1 M1 A1: They have the correct form of the partial fraction and find the correct values of the constants

A1: Correct integration.

M1 B1: They combine their log terms and divides to find the dominant term leading them the find the correct upper limit.

A0: The question requires the answer to be given in the form ln*k* . Here the candidate gives their answer as 2ln (2/3) instead of ln(4/9)

Exemplar Question 3

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3.

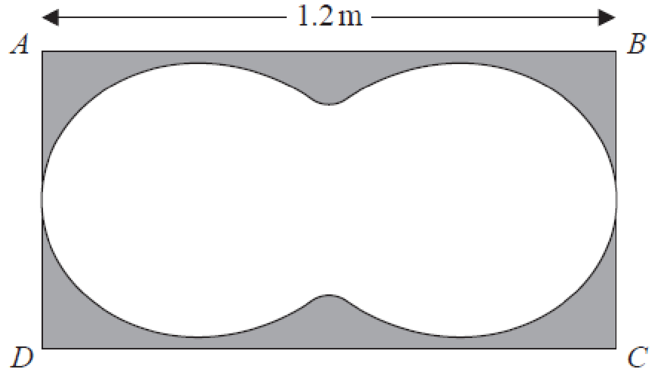

Figure 1

Figure 1 shows the design for a table top in the shape of a rectangle $ABCD$. The length of the table, AB, is 1.2 m. The area inside the closed curve is made of glass and the surrounding area, shown shaded in Figure 1, is made of wood.

The perimeter of the glass is modelled by the curve with polar equation

$$r = 0.4 + a\cos 2\theta \qquad 0 \leqslant \theta < 2\pi$$

where a is a constant.

(a) Show that $a = 0.2$

(2)

Hence, given that $AD = 60$ cm,

(b) find the area of the wooden part of the table top, giving your answer in m^2 to 3 significant figures.

(8)

(Total for Question 3 is 10 marks)

Mean Score 8.2 out of 10

Examiner Comments

This question is assessing polar coordinates and finding the area enclosed by the polar curve (spec ref 7.1, 7.3)

This question proved to be accessible to most candidates. The majority of candidates gained full marks in part (a). One or two lost both marks for using $\theta = 2\pi$ which is not a valid angle for the model.

Part (b) Almost all candidates realised they needed to use the correct area formula and then went on to expand the expression for r. Candidates appeared to be familiar with the method needed for integrating $\cos^2 x$ and most used a fully correct double angle formula in their integral. One or two attempted the double angle formula but then substituted an expression in terms of 2θ rather than 4θ and so lost the second M mark. The majority integrated their expression correctly, although one or two made sign errors or slips when finding the coefficients. Most candidates then went on to substitute the correct limits for their integral. Almost all candidates gained the B1 mark for correctly finding the area of the table top (the mixed of units was dealt with), and most realised they needed to subtract their area enclosed by the curve from the area of the rectangle.
Although most candidates were able to make good progress on this question, sign or arithmetic errors were often made along the way resulting in the A marks being lost.

Mark Scheme

Question	Scheme	Marks	AOs
3(a)(i)	$2(0.4+a)=1.2$ or $0.4+a=0.6$ or $0.4+a\cos 0=0.6$ $\Rightarrow a=...$	M1	3.4
	a = 0.2 * cso	A1*	1.1b
		(2)	
(b)	Area of rectangle is $1.2\times 0.6(=0.72)$	B1	1.1b
	Area enclosed by curve = $\frac{1}{2}\int(0.4+0.2\cos 2\theta)^2(\mathrm{d}\theta)$	M1	3.1a
	$(0.4+0.2\cos 2\theta)^2=0.16+0.16\cos 2\theta+0.04\cos^2 2\theta$ $=0.16+0.16\cos 2\theta+0.04\left(\frac{\cos 4\theta+1}{2}\right)$	M1	2.1
	$\frac{1}{2}\int(0.4+0.2\cos 2\theta)^2\,\mathrm{d}\theta=\frac{1}{2}\left[0.18\theta+0.08\sin 2\theta+0.005\sin 4\theta(+c)\right]$ $=0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta(+c)$ o.e.	A1ft	1.1b
	Area enclosed by curve = $\left[0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta\right]_0^{2\pi}$ or Area enclosed by curve = $2\left[0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta\right]_0^{\pi}$ or Area enclosed by curve = $4\left[0.09\theta+0.04\sin 2\theta+0.0025\sin 4\theta\right]_0^{\pi/2}$	dM1	3.1a
	$=\frac{9}{50}\pi$ or $0.18\pi(=0.5654...)$	A1	1.1b
	Area of wood $=1.2\times 0.6-0.18\pi$	M1	1.1b
	= awrt 0.155 (m^2) cso	A1	1.1b
		(8)	
			(10 marks)

Notes

(a)

M1: Interprets the information from the model and realises that the maximum value of r gives half the length of the table top (or equivalent) and solves to find a value for a. Use $\theta=0$ and $r=0.6$ or $\theta=\pi$ and $r=0.6$ to find a value for a.

Using $\theta=2\pi$ is M0

A1*: Correct value for a.

Alternative

M1: Uses a = 0.2 and $\theta=0$ to find a value for r

A1: Finds r = 0.6 and concludes that a = 0.2

(b)
B1: 1.2×0.6 or 0.72
M1: A correct strategy identified for finding an area enclosed by the polar curve using a correct formula with r substituted. Attempt at area $= \frac{1}{2}\int (0.4+0.2\cos 2\theta)^2 \, d\theta = ...$

Look for $= \lambda \times \frac{1}{2}\int (0.4+0.2\cos 2\theta)^2 \, d\theta = ...$

If the $\frac{1}{2}$ is not explicitly seen then look at the limits and it must be either

$$= \int_0^{\pi} (0.4+0.2\cos 2\theta)^2 \, d\theta = ... \text{ or } = 2\int_0^{\frac{\pi}{2}} (0.4+0.2\cos 2\theta)^2 \, d\theta = ...$$

Condone missing $d\theta$

M1: Squares to achieve three terms and uses $\cos^2 2\theta = \frac{\pm 1 \pm \cos 4\theta}{2}$ to obtain an expression in an integrable form.
A1ft: Correct follow through integration as long as the previous two method marks have been awarded.
dM1: Dependent of first method mark. Finds the required area enclosed by the curve using the correct limits.
There are only three cases either $\frac{1}{2}\int_0^{2\pi} (0.4+0.2\cos 2\theta)^2 \, d\theta$ or $\int_0^{\pi} (0.4+0.2\cos 2\theta)^2 \, d\theta$ or $2\int_0^{\frac{\pi}{2}} (0.4+0.2\cos 2\theta)^2 \, d\theta$

The use of the limit 0 can be implied if it gives 0 but the use of 0 must been seen or implied if it does not result in 0 (just writing 0 is insufficient)
A1: Correct area of the glass following fully correct working. **Do not award for the correct answer following incorrect working.**
M1: Subtracts their area of the glass from their area of the rectangle, as long as it does not give a negative area
A1: awrt 0.155 or awrt $0.155\,\text{m}^2$ (If the units are stated they must be correct) cso

Note: Using a calculator to find the area scores a maximum of B1M1M0A0M0A0M1A1

Student Response A

$\theta = 0, \quad r = 0.6$

$0.4 + a\cos 0 = 0.6$

$0.4 + 1 \times a = 0.6$

$a = 0.2$

b) $1.2 \times 0.6 = 0.72\text{m}^2$

$r = 0.4 + 0.2\cos 2\theta$

$r^2 = 0.16 + 0.16\cos 2\theta + 0.04\cos^2 2\theta$

$= 0.16(0.25\cos^2 2\theta + \cos 2\theta + 1)$

$A = \frac{2}{25}\int \frac{1}{4}\cos^2 2\theta + \cos 2\theta + 1 \, d\theta$

$= \frac{2}{25}\left[\qquad \frac{1}{2}\cos 2\theta + \theta \right.$

$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

$\int \cos 2\theta \cos 2\theta \, d\theta =$

$u = \cos 2\theta \qquad \frac{dv}{dx} = \cos 2\theta$

$u' = 2\cos 2\theta \qquad v = \frac{1}{2}\cos 2\theta$

$= \frac{2}{25}\left[\frac{1}{2}\cos 4\pi + 2\pi \right] = 0.54265$

~~ans = 0.42~~ $0.72 - \text{ans} = 0.177\text{ m}^2$

5/10

Examiner Comments

In part (a)
M1: Uses $\theta = 0$ and $r = 0.6$ to find a value for a
A1: Correct value for a

In part (b)
B1: For the correct area of the rectangle 1.2×1.6
M1: For using the correct formula for the area enclosed by the polar curve
M0: They do NOT use the identity $\cos 4x = 2\cos^2 2x - 1$
A0ft: The mark is only accessible if the previous two method marks have been scored.
dM0: This mark is dependent on the first method mark been scored. The first method mark is scored however the candidate should be using limits of 2π and 0. There is no evidence of using the lower limit of 0.
A0: Incorrect area enclosed by the polar curve.
M1: Finds area rectangle – area enclosed by the polar curve.
A0: Incorrect answer.

Student Response B

$r = 0.4 + a\cos 2\theta$

a) when $a = 0.2$

$r = 0.4 + 0.2\cos(2\theta)$

$= 0.6$, which is half the ~~length~~ perimeter

so $a = 0.6$

b) area of wooden part = area of ~~rectangle~~ rectangle − area of curve

area of rectangle = 6×1.2

$= 7.2\,m^2$

area of curve $= \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta$

$r = 0.4 + 0.2\cos 2\theta$

$r^2 = (0.4 + 0.2\cos 2\theta)(0.4 + 0.2\cos 2\theta)$

$r^2 = ($~~0.16~~ $0.16 + 0.16\cos 2\theta + 0.04\cos^2 2\theta)$

area of curve $= \frac{1}{2}\int_0^{2\pi} 0.16 + 0.16\cos 2\theta + 0.04\cos^2 2\theta \, d\theta$

$= \frac{1}{50}\int_0^{2\pi} 4 + 4\cos 2\theta + \cos^2 2\theta \, d\theta)$

~~$\cos^2 2\theta$~~

~~$\cos 2$~~ $\cos 4\theta = 2\cos^2 2\theta - 1$

$\cos^2 2\theta = \frac{\cos 4\theta + 1}{2}$

$= \frac{1}{50}\int_0^{2\pi} 4 + 4\cos 2\theta + \frac{\cos 4\theta + 1}{2} \, d\theta)$

$= \frac{1}{50}\left[4\theta + 2\sin 2\theta + \frac{1}{8}\sin 4\theta + \frac{\theta}{2}\right]_0^{2\pi}$

$= \frac{1}{50}\left[\frac{9}{2}\theta + 2\sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{2\pi}$

$= \frac{1}{50}(9\pi + 0 + 0) - (0 + 0 + 0)$

$= \frac{9}{50}\pi$

7/10

Examiner Comments

In part (a)
M1: Uses $a = 0.2$ and $\theta = 0$ find a value for r
A0: Incorrect conclusion, should be therefore $a = 0.2$

In part (b)
B0: Incorrect expression for the area of the rectangle. (They use 6×1.2 check the units)
M1: Attempts the correct formula ½ integral r^2 to find an area enclosed by the polar curve.
M1: Squares to achieve three terms and uses ½ $(1 + \cos 4\theta)$ to obtain an integrable form
A1ft: Correct integration
dM1: This mark is dependent on the first method mark been scored. They use the limits 0 and π and twice ½ integral r^2 to give the required area
A1: Correct area of the glass
M1: Subtracts their area of the glass from their area of the rectangle.
A0: Incorrect answer

Student Response C

a) at $\theta = 0$, $r = 0.6$m:

$r = 0.4 + a\cos(2(0))$

$0.6 = 0.4 + a\cos 0$

$0.6 = 0.4 + a$

$0.2 = a$

b) $r = 0.4 + a\cos 2\theta$.

Area of sector: $\frac{1}{2}\int r^2 \, d\theta$.

$\therefore \int_0^{\pi} r^2 \, d\theta$

Area of wood => whole area $- \int_0^{\pi} r^2 d\theta$

A of wood: $0.72 - \int_0^{\pi} (0.4 + 0.2\cos 2\theta)^2 \, d\theta$.

$\Downarrow$

$\int_0^{\pi} (0.16 + 0.16\cos 2\theta + 0.04\cos^2 2\theta) \, d\theta$.

$\cos 2\theta = \cos^2\theta - \sin^2\theta$.

$\cos 2\theta + \sin^2\theta = \cos^2\theta$

$\cos 2\theta + 1 - \cos^2\theta = \cos^2\theta$

$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$.

$\cos^2\theta + \sin^2\theta = 1$.

$\int_0^{\pi} (0.16 + 0.16\cos 2\theta + 0.02(\cos 4\theta + 1)) \, d\theta$

$= \int_0^{\pi} (0.16 + 0.16\cos 2\theta + 0.02\cos 4\theta + 0.02) \, d\theta$

$= [0.18\theta + 0.08\sin 2\theta + 0.005\sin 4\theta]_0^{\pi}$

$= (0.18\pi + 0.08\sin 2\pi + 0.005\sin 4\pi) - 0$.

$= 0.18\pi$

A of wood: $0.72 - 0.18\pi = 0.155$

10/10

Examiner Comments

In part (a)
M1: Uses $\theta = 0$ and $r = 0.6$ to find a value for a
A1: Correct value for a

In part (b)
B1: Correct area of the rectangle 0.72
M1: Attempts the correct formula to find an area enclosed by the polar curve. They are using limits of 0 and π with integral of r^2
M1: Squares to achieve three terms and uses ½ $(1 + \cos 4\theta)$ to obtain an expression in an integrable form.
A1ft: Correct integration
dM1: This mark is dependent on the first method mark been scored. They use the limits 0 and π and twice ½ integral r^2 to give the required area
A1: Correct area of the glass
M1: Subtracts their area of the glass from their area of the rectangle.
A1: Correct answer (condone missing units).

Exemplar Question 4

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4. Prove that, for $n \in \mathbb{Z}, n \geqslant 0$

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

where a, b and c are integers to be found.

(5)

(Total for Question 4 is 5 marks)

Mean Score 2.3 out of 5

Examiner Comments

This question is assessing using the sum of differences for summation of series including use of partial fractions (spec ref 4.4)

This question differentiated well and many candidates found it challenging.
Most candidates realised they needed to split the fraction into partial fractions, and most found correct values for the constants. Most then realised they needed to apply the method of differences. An extremely common error at this point was to start with $r = 1$, omitting $r = 0$, and thus only gaining a maximum of two marks for this question. Some candidates struggled to see how the fractions were cancelling and gave up before considering $r = n – 1$ and $r = n$, and so could only gain the first M1. Candidates should be encouraged to set out a sufficient number of terms in a clear list and indicating clearly which terms remained about differencing, as candidates who did this tended to make better progress. When candidates had algebraic terms they were generally successful in combining them with a correct common denominator. Errors were sometimes made in simplifying the numerator. A few struggled to combine the numerical fraction, and errors were much more common where the candidate didn't attempt to simplify fractions or where they keep 3 algebraic fractions and did not combine $\frac{1}{2(n+2)} - \frac{1}{(n+2)}$

Mark Scheme

Question	Scheme	Marks	AOs
4	$\frac{1}{(r+1)(r+2)(r+3)} \equiv \frac{A}{r+1}+\frac{B}{r+2}+\frac{C}{r+3} \Rightarrow A=\ldots,\ B=\ldots,\ C=\ldots$ $\left(\text{NB } A=\frac{1}{2}\quad B=-1\quad C=\frac{1}{2}\right)$	M1	3.1a
	$r=0$ $\quad \frac{1}{2}\left[\frac{1}{1}-\frac{2}{2}+\frac{1}{3}\right]$ or $\frac{1}{2(1)}-\frac{1}{2}+\frac{1}{2(3)}$ or $\frac{1}{2}-\frac{1}{2}+\frac{1}{6}$ $r=1$ $\quad \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]$ or $\frac{1}{2(2)}-\frac{1}{3}+\frac{1}{2(4)}$ or $\frac{1}{4}-\frac{1}{3}+\frac{1}{8}$ $r=n-1$ $\quad \frac{1}{2}\left[\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}\right]$ or $\frac{1}{2(n)}-\frac{1}{n+1}+\frac{1}{2(n+2)}$ or $\frac{1}{2n}-\frac{1}{n+1}+\frac{1}{2n+4}$ $r=n$ $\quad \frac{1}{2}\left[\frac{1}{n+1}-\frac{2}{n+2}+\frac{1}{n+3}\right]$ or $\frac{1}{2(n+1)}-\frac{1}{n+2}+\frac{1}{2(n+3)}$ or $\frac{1}{2n+2}-\frac{1}{n+2}+\frac{1}{2n+6}$	M1	2.1
	Alternative for this method mark Splits up into $\frac{1}{2}\left(\frac{1}{r+1}\right)-\frac{1}{2}\left(\frac{1}{r+2}\right)+\frac{1}{2}\left(\frac{1}{r+3}\right)-\frac{1}{2}\left(\frac{1}{r+2}\right)$ $r=0$ $\quad \frac{1}{2}\left(\frac{1}{1}\right)-\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{3}\right)-\frac{1}{2}\left(\frac{1}{2}\right)$ or $\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{4}$ $r=1$ $\quad \frac{1}{2}\left(\frac{1}{2}\right)-\frac{1}{2}\left(\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{4}\right)-\frac{1}{2}\left(\frac{1}{3}\right)$ or $\frac{1}{4}-\frac{1}{6}+\frac{1}{8}-\frac{1}{6}$ $r=n$ $\quad \frac{1}{2}\left(\frac{1}{n+1}\right)-\frac{1}{2}\left(\frac{1}{n+2}\right)+\frac{1}{2}\left(\frac{1}{n+3}\right)-\frac{1}{2}\left(\frac{1}{n+2}\right)$	M1	2.1
	$\frac{1}{2}-\frac{1}{2}+\frac{1}{4}+\frac{1}{2(n+2)}-\frac{1}{n+2}+\frac{1}{2(n+3)}$ or $\frac{1}{4}-\frac{1}{2(n+2)}+\frac{1}{2(n+3)}$	A1	1.1b
	$=\frac{n^2+5n+6+2n+6-4n-12+2n+4}{4(n+2)(n+3)}$	M1	1.1b
	$=\frac{(n+1)(n+4)}{4(n+2)(n+3)}$	A1	2.2a
		(5)	
		(5 marks)	

Notes
M1: A complete strategy to find *A*, *B* and *C* e.g. partial fractions. Allow slip when finding the constant but must be the correct form of partial fractions and correct identity. M1: Starts the process of differences to identify the relevant fractions at the start and end. Must have attempted a minimum of $r=0,\quad r=1,\quad \ldots\quad r=n-1$ and $r=n$ Follow through on their values of *A*, *B* and C. Look for $r=0\rightarrow \frac{A}{1}-\frac{B}{2}+\frac{C}{3}$ $\quad r=1\rightarrow \frac{A}{2}-\frac{B}{3}+\frac{C}{4}$ $r=n-1\rightarrow \frac{A}{n}-\frac{B}{n+1}+\frac{C}{n+2}$ $\quad r=n\rightarrow \frac{A}{n+1}-\frac{B}{n+2}+\frac{C}{n+3}$ **<u>Alternative method mark</u>** M1: If they split into $\frac{1}{2}\left(\frac{1}{r+1}\right)-\frac{1}{2}\left(\frac{1}{r+2}\right)+\frac{1}{2}\left(\frac{1}{r+3}\right)-\frac{1}{2}\left(\frac{1}{r+2}\right)$ they only need to find $r=0,\quad r=1,\quad \ldots$ and $r=n$ A1: Correct fractions from the beginning and end that do not cancel stated. M1 Combines all 'their' fractions (at least two algebraic fractions) over their correct common denominator, does not need to be the lowest common denominator (allow a slip in the numerator). A1: Correct answer. **Note**: if they start with $r=1$ the maximum they can score is M1M0A0M1A0 **Note:** Proof by induction gains no marks

Student Response A

$$\frac{1}{(r+1)(r+2)(r+3)} \equiv \frac{A}{r+1} + \frac{B}{r+2} + \frac{C}{r+3}$$

$$1 \equiv A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2)$$

$$r=-2: \quad 1 = -B \Rightarrow B = -1$$

$$r=-3: \quad 1 = 2C \Rightarrow C = \frac{1}{2}$$

$$r=-1: \quad 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$\frac{1}{(r+1)(r+2)(r+3)} \equiv \frac{1}{2(r+1)} - \frac{1}{r+2} + \frac{1}{2(r+3)}$$

$$\sum_{r=0}^{n} \frac{1}{2(r+1)} - \frac{1}{(r+2)} + \frac{1}{2(r+3)}$$

$$= \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{4} + \frac{1}{10}\right)$$

$$+ \left(\frac{1}{8} - \frac{1}{5} + \frac{1}{12}\right) + \ldots$$

$$=$$

When $n=1$,

$$\sum_{r=0}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{(1)(2)(3)} + \frac{1}{(2)(3)(4)} = \frac{1}{6} + \frac{1}{24} = \frac{5}{24}$$

$$\frac{5}{24} = \frac{(n+a)(n+b)}{c(n+2)(n+3)}$$

1/5

Examiner Comments

M1: Correct partial fraction form and finds the values of the constants

M0 A0: Does not find terms for $r = n - 1$ and $r = n$

M0: Does not have any algebraic fractions to combine

A0: Incorrect answer

Student Response B

$$= \frac{A}{r+1} + \frac{B}{r+2} + \frac{C}{r+3}$$

$$A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2) =$$

$r = -2 \quad -B = 1 \quad \boxed{B = -1}$

$r = -3 \quad 2C = 1 \quad \boxed{C = \frac{1}{2}}$

$r = -1 \quad 2A = 1 \quad \boxed{A = \frac{1}{2}}$

$$= \frac{1}{2}\left(\frac{1}{r+1} - \frac{2}{r+2} + \frac{1}{r+3}\right)$$

$(r=1) \quad \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$

$(r=2) \quad \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$

$(r=3) \quad \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$

$(r=4) \quad \frac{1}{5} - \frac{2}{7} + \frac{1}{8}$

$r = (n-1) \quad \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$

$r = n \quad \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3} - \frac{1}{n+2} + \frac{1}{n+3}\right)$$

$$= \frac{1}{2}\left(\frac{1}{6} - \frac{n+3+n+2}{(n+2)(n+3)}\right)$$

$$\frac{1}{2}\left(\frac{1}{6} + \frac{n+2-n-3}{(n+2)(n+3)}\right)$$

$$\frac{1}{2}\left(\frac{1}{6} - \frac{1}{(n+2)(n+3)}\right)$$

$$= \frac{1}{2}\left(\frac{(n+2)(n+3)-6}{6(n+2)(n+3)}\right)$$

$$= \frac{n^2+5n+6-6}{12(n+2)(n+3)}$$

$$= \frac{(n+0)(n+5)}{12(n+2)(n+3)}$$

2/5

Examiner Comments

M1: Correct partial fraction form and finds the values of the constants

M0 A0: Does not start with $r = 0$ which was very common

M1: Combines all their fractions using a common denominator.

A0: Incorrect answer

Student Response C

(5)

$$\frac{1}{(r+1)(r+2)(r+3)} = \frac{A}{r+1} + \frac{B}{r+2} + \frac{C}{r+3}$$

$$1 = A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2)$$

$$r^2+5r+6 \qquad r^2+4r+3 \qquad r^2+3r+2$$

$A + B + C = 0$

$5A + 4B + 3C = 0$ $\qquad A = \frac{1}{2} \quad B = -1 \quad C = \frac{1}{2}$

$6A + 3B + 2C = 1$

$$\frac{1}{2(r+1)} - \frac{1}{r+2} + \frac{1}{2(r+3)}$$

$r=0: \quad \frac{1}{2} - \frac{1}{2} + \frac{1}{6}$

$r=1: \quad \frac{1}{4} - \frac{1}{3} + \frac{1}{8}$

$r=2: \quad \frac{1}{6} - \frac{1}{4} + \frac{1}{10}$

$r=3: \quad \frac{1}{8} - \frac{1}{5} + \frac{1}{12}$

$r=4: \quad \frac{1}{10} - \frac{1}{6} + \frac{1}{14}$

$r=n-1: \quad \frac{1}{2(n)} - \frac{1}{n+1} + \frac{1}{2(n+2)}$

$r=n: \quad \frac{1}{2(n+1)} - \frac{1}{n+2} + \frac{1}{2(n+3)}$

$$\frac{1}{2(n+2)} - \frac{1}{n+2} + \frac{1}{2(n+3)} + \frac{1}{4}$$

$$\frac{4(n+3) - 4(n+3) + 2(n+2) + (n+2)(n+3)}{4(n+2)(n+3)}$$

$$4n+12 - 4n - 12 + 2n+4 + n^2+5n+6$$

$$\frac{n^2+7n+10}{4(n+2)(n+3)} = \frac{(n+5)(n+2)}{4(n+2)(n+3)}$$

$a = 5 \quad b = 2 \quad c = 4$

4/5

Examiner Comments

M1: Correct partial fraction form and finds the values of the constants

M1: Finds a minimum of $r = 0$, $r = 1$, $r = n - 1$ and $r = $ n

A1: Correct non-cancelling terms

M1: Combines all their fractions (at least two algebraic) using a common denominator. Condone the slip in the numerator of the first fraction.

A0: Incorrect answer, following numerical slips when combining fractions.

Exemplar Question 5

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5. A tank at a chemical plant has a capacity of 250 litres. The tank initially contains 100 litres of pure water.

Salt water enters the tank at a rate of 3 litres every minute. Each litre of salt water entering the tank contains 1 gram of salt.

It is assumed that the salt water mixes instantly with the contents of the tank upon entry.

At the instant when the salt water begins to enter the tank, a valve is opened at the bottom of the tank and the solution in the tank flows out at a rate of 2 litres per minute.

Given that there are S grams of salt in the tank after t minutes,

(a) show that the situation can be modelled by the differential equation

$$\frac{dS}{dt} = 3 - \frac{2S}{100 + t}$$

(4)

(b) Hence find the number of grams of salt in the tank after 10 minutes.

(5)

When the concentration of salt in the tank reaches 0.9 grams per litre, the valve at the bottom of the tank must be closed.

(c) Find, to the nearest minute, when the valve would need to be closed.

(3)

(d) Evaluate the model.

(1)

(Total for Question 5 is 13 marks)

Mean Score 6.3 out of 13

Examiner Comments

This question is assessing first order differential equation, including forming equation and integrating factor (spec ref 9.1, 9.2, 9.3)

In this question, part (b) and (d) were answered very well by the majority of candidates, but significantly fewer managed to answer (a) and (c) successfully.

In part (a) many candidates struggled to explain the model successfully. References to the context of the model were required by using words such as "salt in", "volume" and "concentration". There was also some confusion between the salt and salt water that entered the tank. There seemed to be a reluctance by many students to use words as opposed to symbols and they need to practise these explanation skills. There were some excellent fully coherent explanations, but these were relatively rare.
The most common correct explanations are volume = 100 + 3t – 2t = 100 + t and that the rate of salt in = 3. Whilst showing where $\frac{2S}{100+t}$ came from was very poor with candidates just writing $2 \times \frac{S}{100+t}$ with no reasoning where they have come from.

Part (b) was generally attempted correctly with the vast majority of students recognising the type of differential equation and correctly using the integrating factor. A small number of candidates omitted the constant of integration and lost the subsequent marks.
Common errors when attempting to find the constant of integration are using S = 100 or using $0 = 100 + 0 + \frac{c}{100}$ instead of $0 = 100 + 0 + \frac{c}{100^2}$ or achieving c = +1000000 instead of c = -1000000.

Most candidates made an attempt at part (c) although only the minority identified the concentration of salt as the mass of salt divided by volume. The units of grams per litre stated in the question could have been used to guide those who were unsure of this relationship. Successful candidates then generally used their calculators efficiently to solve the resulting cubic equation and find the value of t. Common incorrect approaches were setting S or even $\frac{dS}{dt}$ equal to the value for concentration.

Part (d) was often answered correctly with the most common answer relating to the fact that the mixing would not occur instantly. Many candidates tried to comment on the volume of water or the amount of salt tending to infinity, but did not explain how this contradicted the model. Some candidates noted that the capacity of the tank was 250 litres but did not always highlight that the model would become invalid once the tank was full.

Mark Scheme

Question	Scheme	Marks	AOs
5(a)	The tank initially contains 100L. 3 L are entering every minute and 2 L are leaving every minute so overall 1 L increase in volume each minute so the tank contains 100 + t litres after t minutes	M1	3.3
	2 L leave the tank each minute and if there are Sg of salt in the tank, the concentration will be $\frac{S}{100+t}\,g/L$ so salt leaves the tank at a rate of $2\times\frac{S}{100+t}\,g$ per minute	M1	3.3
	Salt enters the tank at a rate of $3\times 1g$ per minute	B1	2.2a
	$\therefore \frac{dS}{dt}=3-\frac{2S}{100+t}$ * cso	A1*	1.1b
		(4)	
(b)	$\frac{dS}{dt}+\frac{2S}{100+t}=3$		
	$I=e^{\int\frac{2}{100+t}dt}=(100+t)^2 \Rightarrow S(100+t)^2=\int 3(100+t)^2 dt$	M1	3.1b
	$S(100+t)^2=(100+t)^3\,(+c)$ OR $S(100+t)^2=30\,000t+300t^2+t^3\,(+c)$	A1	1.1b
	$t=0,\ S=0 \Rightarrow c=-10^6$	M1	3.4
	$t=10 \Rightarrow S=100+10-\frac{10^6}{(100+10)^2}$ OR $S(100+10)^2=(100+10)^3(+c)\Rightarrow S=...$	dM1	1.1b
	$=$ awrt 27 (g) or $\frac{3310}{121}$ (g)	A1	2.2b
		(5)	
(c)	Concentration is $\left(100+t-\frac{10^6}{(100+t)^2}\right)\div(100+t)=0.9$ OR $S=0.9(100+t)\Rightarrow 0.9(100+t)=(100+t)-\frac{10^6}{(100+10)^2}$ OR $S=0.9(100+t)\Rightarrow 0.9(100+t)^3=(100+t)^3-10^6$	M1	3.4
	$(100+t)^3=10^7\Rightarrow t=...$ OR $t^3+300t^2+30\,000t-9\,000\,000=0\Rightarrow t=...$	dM1	1.1b
	$t=$ awrt 115 (minutes)	A1	2.2b
		(3)	

(d)	E.g. • It is unlikely that mixing is instantaneous • The model will only be valid when the tank is not full • When the valve is closed, the model is not valid • It is unlikely that the concentration of salt water entering the tank remains exactly the same	B1	3.5a
		(1)	
			(13 marks)

Notes

(a)

M1: A suitable explanation for the "100 + *t*" e.g. as a minimum (*v*) = 100 + 3*t* – 2*t* = 100 + *t*

M1: A suitable explanation for the $\frac{2S}{100+t}$

There need to be some explanation (words) for this part of the formula.

e.g. the concentration of (salt) $=\frac{S}{100+t}$ therefore (salt) out $=2\times\frac{S}{100+t}=\frac{2S}{100+t}$

e.g. salt out = $\frac{2S}{\text{volume of water}}=\frac{2S}{100+t}$

Note: M0 for $2\times\frac{S}{100+t}=\frac{2S}{100+t}$ only with no explanation

B1: Correct interpretation for the "3" e.g. salt in = 3 or $\frac{dS}{dt}$ in $=3$

Note: Salt **water** in = 3 is **B0**

A1*: Puts all the components together to form the given differential equation cso

(b)

M1: Uses the model to find the integrating factor and attempts the solution of the differential equation. Look for $I.F.=e^{\int\frac{2}{100+t}dt}\Rightarrow S\times\text{'their }I.F.\text{'}=\int 3\times\text{'their }I.F.\text{' }dt$

A1: Correct solution condone missing + *c*

For the next three mark there must be a constant of integration

M1: Interprets the initial conditions, $t=0\ \ S=0$, and uses in their equation to find the constant of integration.

dM1: Dependent on having a constant of integration. Uses their solution to the problem to find the amount of salt after 10 minutes.

A1: Awrt 27 or $\frac{3310}{121}$. (If the units are stated they must be correct)

Note: If achieves $S(100+t)^2=30\,000t+300t^2+t^3+c$ the constant of integration *c* = 0 and the correct amount of salt can be achieved. If there is no + *c* the maximum they can score is M1A1M0M0A0

(c)

Note: Look out for setting *S* = 0.9 in this part, which scores no marks.

M1: Uses their solution to the model and divides by 100 + *t* as an interpretation of the concentration and sets = 0.9.

Alternatively recognises that the amount of salt = 0.9(100 + *t*) and substitutes for *S* in their solution to the model.

dM1: Dependent on previous method mark. Solves their equation to obtain a value for t. May use a calculator.
A1: Awrt 115 (If the units are stated they must be correct) or 1hr 45 mins with units
(d)
B1: Evaluates the model by making a suitable comment – see scheme for examples.

Student Response A

a) ~~Fo~~ Vol total = ~~250~~ 100 + 3t − 2t

= 100 − t

Salt in = 3×1 = 3g

Salt out = ~~1×(100−2t)~~ $1 \times \frac{2S}{100-t}$

$$\frac{ds}{dt} = 3 - \frac{2s}{100-t}$$

b) $(100+t)\frac{ds}{dt} = 3(100+t) - 2S$

$(100+t)\frac{ds}{dt} + 2s = 3(100+t)$

$e^{\int 2\,dt} = e^{2t}$

~~(Fo~~ $\frac{d}{dt}(e^{2t} S) = 3e^{2t}(100+t)$

$e^{2t}S = \int$ ~~$3000e^{2t} + te^{2t}$~~ $3e^{2t}(100+t)\,dt$

$V = 3e^{2t}$ $\quad du = 100 + t$

$dv = 6e^{2t}$ $\quad u = 100t + \frac{1}{2}t^2$

$e^{2t}S = \int 3te^{2t}(100 + t^2) + k$ $\int$

$e^{2t}S = \int 3e^{2t} \quad 300e^{2t} + 3te^{2t}$

$e^{2t}S = 150e^{2t} + \text{Cons} + 3\int te^{2t}$

$v = t \quad du = e^{2t}$

$dv = 1 \quad u = \frac{1}{2}e^{2t}$

$3\int te^{2t} = \frac{1}{2}te^{2t} - \int \frac{1}{2}e^{2t}$

$= \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + \text{Cons}$

$e^{2t}S = 150e^{2t} + \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C$

$S = \frac{599}{4} + \frac{1}{2}t + \frac{C}{e^{2t}}$

$S = 0 \quad t = 0$

$0 = \frac{599}{4} + C$

$C = -\frac{599}{4}$

$S = \frac{599}{4} + \frac{1}{2}t - \frac{599}{4e^{2t}}$

let $t = 10$

$S = \frac{599}{4} + 5 - \frac{599}{4e^{20}}$

$= 154.7g$

~~$250 = \frac{599}{4} + \frac{1}{2}t - \frac{599}{4e^{2t}}$~~

~~$\frac{401}{4} = \frac{1}{2}t - \frac{599}{4e^{2t}}$~~

~~Vol = 250~~

$0.9 \times 250 = 225$

$225 = \frac{599}{4} + \frac{1}{2}t - \frac{599}{4e^{2t}}$

$\frac{301}{4} = \frac{1}{2}t - \frac{599}{4e^{2t}}$

$301 = 2t - \frac{599}{e^{2t}}$

$301e^{2t} = 2te^{2t} - 599$

d) may not be accurate as in reality not all salt will dissolve.

4/13

Examiner Comments

In part (a)
M0: States that the volume = 100 + 3t - 2t but does not proceed to 100 + *t* they have 100 - t
M0: Does not explain where $\frac{2S}{100+t}$ comes from, they just state that this is the salt out.
B1: Correct interpretation of the 3 (salt in)
A0: Must have all the previous marks to score the final mark.

In part (b)
M0 A0: Incorrect method to find the integrating factor. They have multiplied by (100 + t) and is therefore of the wrong form.
M1: Uses the initial conditions $t = 0$ and $s = 0$ to find their constant of integration
dM1: Dependent on achieving the previous method mark. Substitutes $t = 10$ into their equation containing a constant and proceeds to S = ...
A0: Incorrect answer

In part (c)
M0: Does not use the concentration of S is 0.9 implies $0.9 = \frac{S}{100+t}$, they use S = 0.9 × 250
dM0 A0: The method mark required the previous mark to be achieved to score any more marks.

In part (d)
B1: Correct evaluation, comments on that the salt will not all dissolve

Student Response B

(1

a) Input of salt = 3 litres/min × 1 gram/litre
= 3 grams / ~~litre~~ min

Output of salt = 2 litres/min × ~~salt~~ current grams / litre

~~salt~~ current grams / litre = S / volume of water = $\frac{S}{100+3t-2t} = \frac{S}{100+t}$

∴ Output of salt = $\frac{2S}{100+t}$ grams / ~~litre~~ min

∴ $\frac{dS}{dt} = 3 - \frac{2S}{100+t}$

b) ~~$\frac{dS}{dt} = \frac{300+3}{100+t}$~~

$\frac{dS}{dt} + \frac{2S}{100+t} = 3$ $\qquad e^{\int \frac{2}{100+t}dt} = e^{2\ln(100+t)}$
$= (100+t)^2$

$(100+t)^2 \frac{dS}{dt} + 2S(100+t) = 3(100+t)^2$

$\frac{d}{dt}\left(S(100+t)^2\right) = 3(100+t)^2$

∴ $S(100+t)^2 = \int 3(100+t)^2 \, dt$

$= \int (30000 + 600t + 3t^2)\, dt$

$S(100+t)^2 = 30000t + 300t^2 + t^3$

$\therefore \; S = \dfrac{t(t^2+300t+30000)}{(100+t)^2}$

At $t=10$, $S = \dfrac{3310}{121}$ or 27.36 grams at 10 minutes

c) $\dfrac{dS}{dt} = \dfrac{(100+t)^2(3t^2+600t+30000) - (30000t+300t^2+t^3)\cdot 2(100+t)}{(100+t)^4}$

$= \dfrac{3(100+t)^3 - 2(30000t+300t^2+t^3)}{(100+t)^3}$

If $\frac{dS}{dt}=0.9$, $\; 0.9 = \dfrac{3(100+t)^3 - 2(30000t+300t^2+t^3)}{(100+t)^3}$

$0.9(100+t)^3 = 3(100+t)^3 - 2t(30000+300t+t^2)$

$2t(30000+300t+t^2) = 2.1(100+t)^3$

$60000t + 600t^2 + 2t^3 = 2.1(1000000 + 30000t + 300t^2 + t^3)$

$0.1t^3 + 30t + 3000t + 2100000 = 0$

$t = -371.4$

d) The assumption that the salt water mixes instantly is a big flaw.

7/13

Examiner Comments

In part (a)

M1: Shows that volume = 100 + 3t - 2t = 100 + t seen within the fraction

M1: Explains where $\frac{2S}{100+t}$ comes from $\frac{\text{2 litres}}{\text{min}}$, $\times$ $\frac{\text{current grams}}{\text{litre}}$, ($\frac{\text{current grams}}{\text{litre}} = \frac{S}{\text{volume}}$ …)

B1: Correct interpretation of the 3, input of salt = 3 (3 litres/min × 1 gram)

A1: Fully correct explanation, all previous marks have been achieved and the printed answer written

In part (b)

M1: Correct integrating factor and attempts to solve the differential equation

A1: Correct general solution

M0: Does not have a constant of integration to find so cannot score this mark. In this form of the general equation the constant of integration is 0 however the candidate needs use the initial conditions to find/state that c = 0.

dM0 A0: As they have not scored the previous method mark and the next method mark is dependent therefore dM0 even though they use t = 10 to find a value for S.

In part (c)

M0: Does not use 0.9 = $\frac{S}{100+t}$, here they mis interpret concentrate for rate of change of S and incorrectly sets $\frac{dS}{dt}$ = 0.9

dM0 A0: The method mark required the previous mark to be achieved to score any more marks

In part (d)

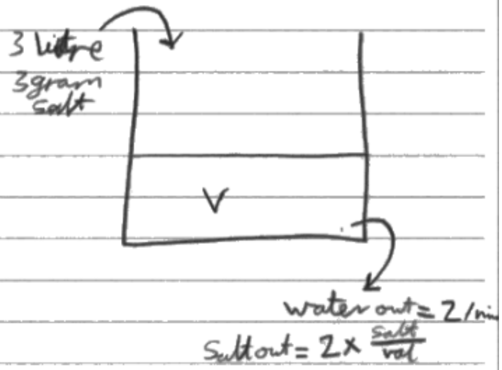
B1: Correct evaluation, comments on salt water missing instantly

Student Response C

$a) \frac{ds}{dt}$ = salt in − salt out

Salt in = 3 g

Salt out = $2 \times \frac{\text{salt}}{\text{vol}}$



vol = $100 + 3t - 2t = 100 + t$

$\therefore$

$$\frac{ds}{dt} = 3 - \frac{2s}{100+t}$$

$b) \frac{ds}{dt} + \frac{2s}{100+t} = 3$

$$IF = e^{\int \frac{2}{100+t} dt} = e^{2\ln(100+t)} = e^{\ln|t^2+200t+1000|} = t^2 + 200t + 1000$$

$$\frac{d}{dt}\left((t^2+200t+1000)s\right) = 3t^2 + 600t + 3000$$

$$(t^2+200t+1000)s = t^3 + 300t^2 + 3000t + c$$

$$s = \frac{t^3 + 300t^2 + 300t + c}{t^2 + 200t + 1000}$$

@ $s = 0$, $t = 0$

$$\therefore 0 = \frac{c}{1000} \Rightarrow c = 0$$

@ $t = 10$,

$$s = \frac{10^3 + 300(10^2) + 300(10)}{10^2 + 200(10) + 1000} \doteq \frac{340}{31} \text{ g}$$

(c) conc = $\frac{S}{vol}$

@ conc = 0.9, $90 + 0.9t = \frac{t^3 + 300t^2 + 3000t}{t^2 + 200t + 1000}$

$t^3 + 300t^2 + 3000t = 0.9t^3 + 180t^2 + 900t + 90t^2 + 18000t + 90\,000$

$0.1t^3 + 30t^2 - 15900t - 90000 = 0$

$t = 280$ min

(d) does not take into account time taken for the salt to diffuse through the solution.

10/13

Examiner Comments

In part (a)
M1: Shows that volume = 100 + 3t - 2t = 100 + t
M1: Explains where $\frac{2S}{100+t}$ comes from, salt in = 2 × $\frac{\text{salt}}{\text{volume}}$
B1: Correct interpretation of the 3, salt in = 3
A1: Fully correct explanation, all previous marks have been achieved and the printed answer written

In part (b)
M1: Correct method to find the integrating factor and attempts to solve the differential equation. There is a numerical slip when multiplying out (t + 100)2
A0: Incorrect general solution due to earlier slip
M1: Uses the initial conditions t = 0 and s = 0 to find their constant of integration
dM1: Dependent on achieving the previous method mark. Substitutes t = 10 into their equation containing a constant and proceeds to S = ...
A0: Incorrect answer

In part (c)
M1: Uses 0.9 = $\frac{S}{100+t}$, here they rearrange to get S = 0.9(100 + t) and sets equal to their answer to part (b).
dM1: The method mark required the previous mark and they solve their equation to find a value for t.
A0: Incorrect value

In part (d)
B1: Correct evaluation, comments on time taken for the salt to diffuse.

Exemplar Question 6

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6. Prove by induction that for all positive integers n

$$f(n) = 3^{2n+4} - 2^{2n}$$

is divisible by 5

(6)

(Total for Question 6 is 6 marks)

Mean Score 4.5 out of 6

Examiner Comments

This question is assessing proof by induction for divisibility (spec ref 1.1)
In this question, the majority of the candidates manage to score the first three marks by showing the basis step, make a correct assumption for $n = k$ and giving an expression for $n = k + 1$, but accuracy marks proved harder to score.
Virtually all candidates seemed familiar with the method required for this type of proof by induction. They tested the $n = 1$ case and concluded that this case was true. Occasionally some candidates failed to draw a conclusion. Again, almost all made an assumption for $n = k$ case and then considered $f(k + 1)$ or $f(k + 1) - f(k)$ (with other appropriate combinations of the two functions seen and used). There were many slips seen in dealing with the powers,
and $3^{2k+6} - 2^{2k+2} = f(k)(3^2-2^2)$ was seen on more than one occasion. Some, having obtained a correct expression involving $f(k + 1)$ and $f(k)$, did not state $f(k+1)$ explicitly was an expression divisible by 5. Of those who completed the algebra, most (but not all) stated a clear and logical conclusion drawing the elements of the proof together. Many candidates dropped the final mark for not stating if true for k **then** true for $k + 1$.

Mark Scheme

Question	Scheme	Marks	AOs
6	**<u>Way 1</u>** f(k +1) – f(k)		
	When n = 1, $3^{2n+4}-2^{2n}=729-4=725$ $(725=145\times 5)$ so the statement is true for n = 1	B1	2.2a
	Assume true for n = k so $3^{2k+4}-2^{2k}$ is divisible by 5	M1	2.4
	$\mathrm{f}\left(k+1\right)-\mathrm{f}\left(k\right)=3^{2k+6}-2^{2k+2}-3^{2k+4}+2^{2k}$	M1	2.1
	either $8\mathrm{f}\ \ k\ +5\times 2^{2k}$ or $3\mathrm{f}\ \ k\ +5\times 3^{2k+4}$	A1	1.1b
	$\mathrm{f}\ \ k+1\ =9\mathrm{f}\ \ k\ +5\times 2^{2k}$ or $\mathrm{f}\ \ k+1\ =4\mathrm{f}\ \ k\ +5\times 3^{2k+4}$ o.e.	A1	1.1b
	<u>If true for n = k then it is true for</u> <u>n = k + 1</u> and as it is <u>true for n = 1</u>, the statement is <u>true for all</u> <u>(positive integers) n</u>. (Allow 'for all values')	A1	2.4
		(6)	
	<u>Way 2</u> f(k +1)		
	When n = 1, $3^{2n+4}-2^{2n}=729-4=725$ $(725=145\times 5)$so the statement is true for n = 1	B1	2.2a
	Assume true for n = k so $3^{2k+4}-2^{2k}$ is divisible by 5	M1	2.4
	$\mathrm{f}\left(k+1\right)=3^{2(k+1)+4}-2^{2(k+1)}\left(=3^{2k+6}-2^{2k+2}\right)$	M1	2.1
	$\mathrm{f}\ \ k+1\ =9\mathrm{f}\ \ k\ +5\times 2^{2k}$ or $\mathrm{f}\ \ k+1\ =4\mathrm{f}\ \ k\ +5\times 3^{2k+4}$ o.e.	A1 A1	1.1b 1.1b
	<u>If true for n = k then it is true for</u> <u>n = k + 1</u> and as it is <u>true for n = 1</u>, the statement is <u>true for all</u> <u>(positive integers) n</u>. (Allow 'for all values')	A1	2.4
		(6)	
	<u>Way 3</u> f(k) = 5M		
	When n = 1, $3^{2n+4}-2^{2n}=729-4=725$ $(725=145\times 5)$ so the statement is true for n = 1	B1	2.2a
	Assume true for n = k so $3^{2k+4}-2^{2k}=5M$	M1	2.4
	$\mathrm{f}\left(k+1\right)=3^{2(k+1)+4}-2^{2(k+1)}\left(=3^{2k+6}-2^{2k+2}\right)$	M1	2.1
	$\left(\mathrm{f}\left(k+1\right)=3^2\times 3^{2k+4}-2^2\times 2^{2k}=3^2\times\left(5M+2^{2k+2}\right)-2^2\times 2^{2k}\right)$ $\mathrm{f}\ \ k+1\ =45M+5\times 2^{2k}$ o.e. OR $\left(\mathrm{f}\left(k+1\right)=3^2\times 3^{2k+4}-2^2\times 2^{2k}=3^2\times 3^{2k+4}-2^2\times\left(3^{2k+4}-5M\right)\right)$ $\mathrm{f}\ \ k+1\ =5\times 3^{2k+4}+20M$ o.e.	A1 A1	1.1b 1.1b
	<u>If true for n = k then it is true for</u> <u>n = k + 1</u> and as it is <u>true for n = 1</u>, the statement is <u>true for all</u> <u>(positive integers) n</u>. (Allow 'for all values')	A1	2.4
		(6)	

	Way 4 f(k +1) + f(k)		
	When n = 1, $3^{2n+4}-2^{2n}=729-4=725$ ($725=145\times5$) so the statement is true for n = 1	B1	2.2a
	Assume true for n = k so $3^{2k+4}-2^{2k}$ is divisible by 5	M1	2.4
	$\text{f}(k+1)+\text{f}(k)=3^{2k+6}-2^{2k+2}+3^{2k+4}-2^{2k}$	M1	2.1
	$\text{f}(k+1)+\text{f}(k)=3^2\times3^{2k+4}-2^2\times2^{2k}+3^{2k+4}-2^{2k}$ leading to $10\times3^{2k+4}-5\times2^{2k}$	A1	1.1b
	$f(k+1)=5[2\times3^{2k+4}-2^{2k}]-f(k)$ o.e.	A1	1.1b
	<u>If true for n = k then it is true for n = k + 1</u> and as it is <u>true for n = 1</u>, the statement is <u>true for all (positive integers) n</u>. (Allow 'for all values')	A1	2.4
		(6)	
	Way 5 f(k +1) – 'M'f(k) (Selecting a value of M that will lead to multiples of 5)		
	When n = 1, $3^{2n+4}-2^{2n}=729-4=725$ ($725=145\times5$) so the statement is true for n = 1	B1	2.2a
	Assume true for n = k so $3^{2k+4}-2^{2k}$ is divisible by 5	M1	2.4
	$\text{f}(k+1)-\text{'M'f}(k)=3^{2k+6}-2^{2k+2}-\text{'M'}\times3^{2k+4}+\text{'M'}\times2^{2k}$	M1	2.1
	$\text{f}(k+1)-\text{'M'f}(k)=(9-\text{'M'})\times3^{2k+4}-(4-\text{'M'})\times2^{2k}$	A1	1.1b
	$\text{f}(k+1)=(9-\text{'M'})\times3^{2k+4}-(4-\text{'M'})\times2^{2k}+\text{'M'f}(k)$ o.e.	A1	1.1b
	<u>If true for n = k then it is true for n = k + 1</u> and as it is <u>true for n = 1</u>, the statement is <u>true for all (positive integers) n</u>. (Allow 'for all values')	A1	2.4
		(6)	

(6 marks)

Notes

<u>Way 1</u> f(k +1) – f(k)

B1: Shows the statement is true for n = 1. Needs to show f(1) = 725 and conclusion true for n = 1, this statement can be recovered in their conclusion if says e.g. true for n = 1

M1: Makes an assumption statement that assumes the result is true for n = k. Assume (true for) $n=k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n=k$ then …etc

M1: Attempts f(k +1) – f(k) or equivalent work

A1: Achieves a correct simplified expression for f(k +1) – f(k)

A1: Achieves a correct expression for f(k +1) in terms of f(k)

A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

<u>Way 2</u> f(k +1)

B1: Shows the statement is true for n = 1. Needs to show f(1) = 725 and conclusion true for n = 1, this statement can be recovered in their conclusion if says e.g. true for n = 1.

M1: Makes an assumption statement that assumes the result is true for n = k. Assume (true for) $n=k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n=k$ then …etc

M1: Attempts f(k +1)
A1: Correctly achieves either $9\text{f}(k)$ **or** 5×2^{2k} or either $4\text{f}(k)$ **or** $5\times3^{2k+4}$
A1: Achieves a correct expression for f(k +1) in terms of f(k)
A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

Way 3 f(k) = 5M
B1: Shows the statement is true for n = 1. Needs to show f(1) = 725 and conclusion true for n = 1, this statement can be recovered in their conclusion if says e.g. true for n = 1.
M1: Makes an assumption statement that assumes the result is true for n = k. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then …etc
M1: Attempts f(k +1)
A1: Correctly achieves either $45M$ **or** 5×2^{2k} or either $20M$ **or** $5\times3^{2k+4}$
A1: Achieves a correct expression for f(k +1) in terms of M and 2^{2k} or M and 3^{2k+4}
A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

Way 4 f(k +1) + f(k)
B1: Shows the statement is true for n = 1. Needs to show f(1) = 725 and conclusion true for n = 1, this statement can be recovered in their conclusion if says e.g. true for n = 1
M1: Makes an assumption statement that assumes the result is true for n = k. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then …etc
M1: Attempts f(k +1) + f(k) or equivalent work
A1: Achieves a correct simplified expression for f(k +1) + f(k)
A1: Achieves a correct expression for $\text{f}(k+1) = 5\left[2\times3^{2k+4} - 2^{2k}\right] - \text{f}(k)$
A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

Way 5 f(k +1) – Mf(k) (Selects a suitable value for M which leads to divisibility of 5)
B1: Shows the statement is true for n = 1. Needs to show f(1) = 725 and conclusion true for n = 1, this statement can be recovered in their conclusion if says e.g. true for n = 1
M1: Makes an assumption statement that assumes the result is true for n = k. Assume (true for) $n = k$ is sufficient. This mark may be recovered in their conclusion if they say e.g. if true for $n = k$ then …etc
M1: Attempts f(k +1) – Mf(k) or equivalent work
A1: Achieves a correct simplified expression, $\text{f}(k+1) - \text{'M'f}(k)$ which is divisible by 5
$\text{f}(k+1) - \text{'M'f}(k) = (9-\text{'M'})\times3^{2k+4} - (4-\text{'M'})\times2^{2k}$
A1: Achieves a correct expression for $\text{f}(k+1) = (9-\text{'M'})\times3^{2k+4} - (4-\text{'M'})\times2^{2k} + \text{'M'f}(k)$ which is divisible by 5
A1: Correct complete conclusion. This mark is dependent on **all** previous marks. It is gained by conveying the ideas of **all** underlined points either at the end of their solution **or** as a narrative in their solution.

Student Response A

$f(1) = 3^{2+4} - 2^{2} = 725$

$725 \div 5 = 145$

true

$f(k+1) = 3^{2k+2+4} - 2^{2k+2}$

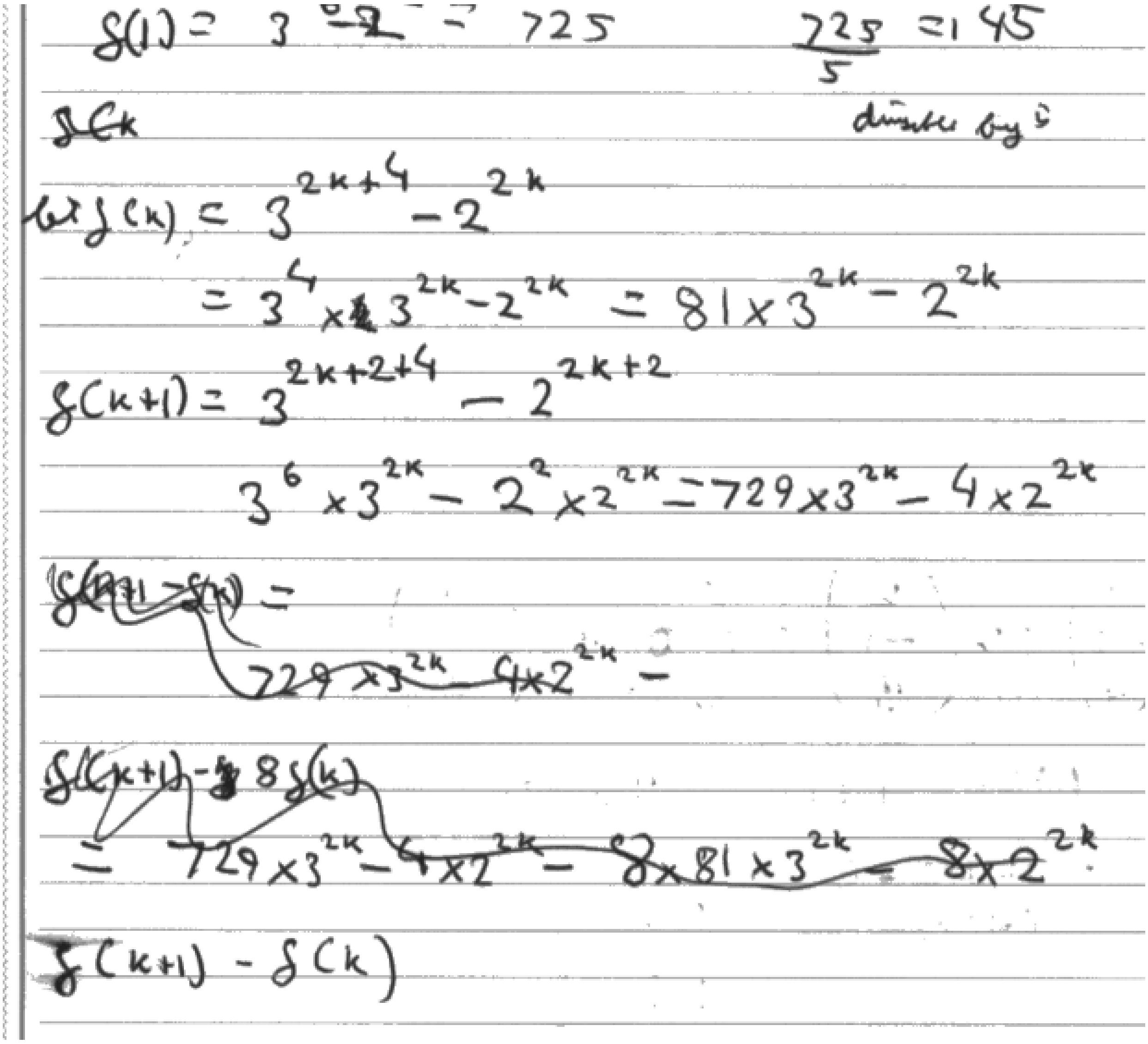
$= 3^{2k+6} - 2^{2k+2}$

$= 3^{6} \times 3^{2k} - 2^{2} \times 2^{2k} = 725 \times 2^{2k}$

~~$f(k+1) - f(k) = 3^{6} \times 3^{2k} - 2^{2} \times 2^{2k} - 3^{4} \times 3^{2k} - 2^{2k}$~~

$729 \times 3^{2k} - 4 \times 2^{2k} - 81 \times 3^{2k} - 2^{2k}$

$725 \times 2^{2k} \div 5 = 145 \times 2^{2k}$



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Examiner Comments

Here this candidate has two attempts which are as both as complete as each other and score the same marks.

Using way 2 on the mark scheme
B1: Shows that the statement is true for $n = 1$, f(1) = 725 and 'true' or 'divisible by 5'
M0: Does not makes the assumption that $n = k$ is true, they have let f(k) which is not sufficient.
M1: Attempts f(k + 1)
A0 A0 A0: Does not achieve a correct term for f(k + 1) so no more marks are available.

Student Response B

$n=1$ ∴

$3^{2+4} - 2^2 = 3^6 - 4 = 729 - 4$

$= 725$

$\frac{725}{5} = 145$

∴ shown to be true for $n=1$

Assume true for $n=k$

$f(k) = 3^{2k+4} - 2^{2k}$ is divisible by 5

When $n=k+1$

$f(k+1) = 3^{2(k+1)} - 2^{2(k+1)} = 3^{2k+2} - 2^{2k+2}$

$= 3^{2k+2} - 4 \cdot 2^{2k}$

$f(k) = 9 \cdot 3^{2k+2} - 2^{2k}$

$f(k) + f(k+1) = 10 \cdot 3^{2k+2} - 5 \cdot 2^{2k}$

$5(2 \cdot 3^{2k+2} - 2^{2k})$

↑ ∴ can be divided by 5

∴ The summation is true for $n=k+1$ if the assumption it is true for $n=k$. As it is shown to be true when $n=1$ it is true for all +ve n integers.

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Examiner Comments

Using way 4 on the mark scheme

B1: Finds that when $n = 1$, f(1) = 725 = 5(145). The concludes that it is true for $n = 1$.

M1: Makes the assumption that $n = k$ is true

M1: Attempts $f(k + 1) + f(k)$

A1: Correct expression for $f(k + 1) + f(k)$

A0: Does not achieve a correct expression for $f(k + 1)$. If they are $f(k + 1) \pm$ multiple of $f(k)$, they need to get to $f(k + 1) = \ldots$ or explain that $f(k + 1)$ is divisible by 5 due to $f(k + 1) \pm$ multiple of f(k) is divisible by 5 and $f(k)$ is divisible by 5.

A0: As previous mark is A0, this mark cannot be scored.

Student Response C

(6)

$f(n) = 3^{2n+4} - 2^{2n}$

let $n = 1$, $f(1) = 3^6 - 2^2$

$= 729 - 4 = 725 = 5(145)$

assume this is true for $n = k$

$f(k) = 3^{2k+4} - 2^{2k}$

let $n = (k+1)$

$3^{2(k+1)+4} - 2^{2(k+1)}$

$= 3^{2k+6} - 2^{2k+2}$

$= 9 \cdot 3^{2k+4} - 4 \cdot 2^{2k}$

$= 5 \cdot 3^{2k+4} + 4(3^{2k+4} - 2^{2k})$

$= 5 \cdot 3^{2k+4} + 4 f k$

$f(k)$ is ÷ by 5 and

$5 \cdot 3^{2k+4} = 5(3^{2k+4})$

therefore $f(k+1)$ is divisible by 5

as it is true for $n = 1$, $n = k$ & $n = k+1$ it is proven by induction to be true for all $n \in \mathbb{Z}^+$

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Examiner Comments

Using way 2 on the mark scheme
B1: Finds that when $n = 1$, $f(1) = 725 = 5(145)$. The conclusion that therefore true for $n = 1$ is true is recovered in the conclusion.
M1: Makes the assumption that $n = k$ is true
M1: Attempts $f(k + 1)$
A1 A1: Achieves a correct expression for $f(k + 1)$
A0: Their conclusion is not sufficient, it does not imply 'if true for $n = k$ then true for $n = k + 1$'

Exemplar Question 7

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7. The line l_1 has equation

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-4}{3}$$

The line l_2 has equation

$$\mathbf{r} = \mathbf{i} + 3\mathbf{k} + t(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

where t is a scalar parameter.

(a) Show that l_1 and l_2 lie in the same plane.

(3)

(b) Write down a vector equation for the plane containing l_1 and l_2

(1)

(c) Find, to the nearest degree, the acute angle between l_1 and l_2

(3)

(Total for Question 7 is 7 marks)

Mean Score 4.1 out of 7

Examiner Comments

This question is assessing vectors, understanding the forms of vector equations of lines, equations of planes and the angle between two lines (spec ref 6.1, 6.2, 6.3)

Part (a) Candidates generally either chose to try to show that the lines intersected at a point or tried to find a vector which was perpendicular to both lines. Those candidates who tried to show that the lines intersected generally set up three equations and tried to solve one pair of them. Some candidates then failed to use their solution in the third equation and a significant number then failed to make a full conclusion. Those candidates who tried to find a perpendicular vector generally managed to do so, typically using the cross product, but then failed to use this with the coordinate points to find the equation of the plane to show that both lines lay on the plane. Many candidates failed to achieve the final mark as they did not give a reason why the lines lie in the same plane i.e. point of intersection

(b) Most candidates gave a correct form for the equation of the plane, but a significant number lost this mark because they failed to write as an equation r = ... (many incorrectly wrote π =…)

(c) Most candidates successfully used the dot product to obtain the correct value for cosθ but some failed to give their answer to the nearest degree or used arcsin instead of arccos.

Mark Scheme

Question	Scheme	Marks	AOs
7(a) **Way 1**	$1+2\lambda=1+t$ $-1-\lambda=-t$ $4+3\lambda=3+2t$ $\Rightarrow t=\ldots \text{or } \lambda=\ldots$	M1	3.1a
	Checks the third equation with $t=2$ and $\lambda=1$ Or shows that the coordinate (3, -2, 7) lies on both lines	A1	1.1b
	As the lines intersect at a point the lines lie in the same plane.	A1	2.4
		(3)	
(a) **Way 2**	$1=1+2\lambda+t$ $-1=\quad-\lambda-t$ $4=3+3\lambda+2t$ $\Rightarrow t=\ldots \text{or } \lambda=\ldots$ $1=1+2\lambda+t$ $0=-1-\lambda-t$ $3=4+3\lambda+2t$ $\Rightarrow t=\ldots \text{or } \lambda=\ldots$	M1	3.1a
	Checks the third equation with $t=2$ and $\lambda=-1$ Checks the third equation with $t=-2$ and $\lambda=1$	A1	1.1b
	Second coordinates lie on the plane; therefore, the lines lie on the same plane	A1	2.4
		(3)	
(a) **Way 3**	$x=1+t,\quad y=-t,\quad z=3+2t$ $\frac{1+t-1}{2}=\frac{-t+1}{-1}=\frac{3+2t-4}{3}$ Solves a pair of equations $t=\ldots$	M1	3.1a
	Solve two pairs of equations to find $t=2$	A1	1.1b
	As the lines intersect at a point the lines lie in the same plane.	A1	2.4
		(3)	
(a) **Way 4 (Using Further Pure 2 knowledge)**	$\begin{pmatrix}2\\-1\\3\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix}\Rightarrow 2x-y+3z=0$ and $\begin{pmatrix}1\\-1\\2\end{pmatrix}.\begin{pmatrix}x\\y\\z\end{pmatrix}\Rightarrow x-y+$ $2z=0$ attempts to solve the equations to find a normal vector OR attempts the cross product $\begin{pmatrix}2\\-1\\3\end{pmatrix}\times\begin{pmatrix}1\\-1\\2\end{pmatrix}=\ldots$ **AND** either finds the equation of one plane **OR** finds dot product between the normal and one coordinate $r.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\begin{pmatrix}1\\-1\\4\end{pmatrix}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\ldots$ or $r.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\begin{pmatrix}1\\0\\3\end{pmatrix}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\ldots$	M1	3.1a

	OR $\begin{pmatrix}1\\-1\\4\end{pmatrix}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\ldots$ or $\begin{pmatrix}1\\0\\3\end{pmatrix}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=\ldots$		
	Achieves the correct planes containing **each** line $r.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=-2$ or $x-y-z=-2$ o.e. **OR** Shows that $\begin{pmatrix}1\\-1\\4\end{pmatrix}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=-2$ **and** $\begin{pmatrix}1\\0\\3\end{pmatrix}.\begin{pmatrix}1\\-1\\-1\end{pmatrix}=-2$ o.e.	A1	1.1b
	Both planes are the same, therefore the lines lie in the same plane.	A1	2.4
		(3)	
(b)	e.g. $\mathbf{r}=\begin{pmatrix}1\\0\\3\end{pmatrix}+p\begin{pmatrix}2\\-1\\3\end{pmatrix}+q\begin{pmatrix}1\\-1\\2\end{pmatrix}$ or $\mathbf{r}=\begin{pmatrix}1\\-1\\4\end{pmatrix}+p\begin{pmatrix}2\\-1\\3\end{pmatrix}+q\begin{pmatrix}1\\-1\\2\end{pmatrix}$ or $\mathbf{r}=\begin{pmatrix}3\\-2\\7\end{pmatrix}+p\begin{pmatrix}2\\-1\\3\end{pmatrix}+q\begin{pmatrix}1\\-1\\2\end{pmatrix}$ or $\mathbf{r}=\begin{pmatrix}3\\-2\\7\end{pmatrix}+p\begin{pmatrix}0\\-1\\1\end{pmatrix}+q\begin{pmatrix}1\\-1\\2\end{pmatrix}$ or $r.k\begin{pmatrix}1\\-1\\-1\end{pmatrix}=-2k$	B1	2.5
		(1)	
(c) **Way 1**	$\begin{pmatrix}2\\-1\\3\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\2\end{pmatrix}=2+1+6$	M1	1.1b
	$\sqrt{2^2+(-1)^2+3^2}\sqrt{1^2+(-1)^2+2^2}\cos\theta=9$ $\Rightarrow\cos\theta=\dfrac{9}{\sqrt{2^2+(-1)^2+3^2}\sqrt{1^2+(-1)^2+2^2}}$	dM1	2.1
	$\theta=11$ cao	A1	1.1b
		(3)	
Way 2 **(Using Further Pure 2 knowledge)**	$\begin{pmatrix}2\\-1\\3\end{pmatrix}\times\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$	M1	1.1b
	$\sqrt{2^2+(-1)^2+3^2}\sqrt{1^2+(-1)^2+2^2}\sin\theta=\sqrt{1^2+(-1)^2+(-1)^2}$	dM1	2.1

	$\Rightarrow \sin\theta = \dfrac{\sqrt{1^2+(-1)^2+(-1)^2}}{\sqrt{2^2+(-1)^2+3^2}\sqrt{1^2+(-1)^2+2^2}}$		
	$\theta = 11$ cao	A1	1.1b
		(3)	
			(7 marks)

Notes

(a)

Allow using $\begin{pmatrix}1\\3\\0\end{pmatrix}$ instead of $\begin{pmatrix}1\\0\\3\end{pmatrix}$ for the method mark.

Way 1

M1: Starts by attempting to find where the two lines intersect. They must set up a parametric equation for line 1 (allow sign slips and as long as the intention is clear), forms simultaneous equations by equating coordinates and attempts to solve to find a value for $t = \ldots$ or $\lambda = \ldots$.

A1: Shows that there is a unique solution by checking the third equation or shows that the coordinate (3, -2, 7) lies on both lines.

A1: Achieves the correct values $t = 2$ and $\lambda = 1$, checks the third equation and concludes that either

- a common point,
- the lines intersect
- the equations are consistent

therefore, the lines lie in the same plane

Way 2

M1: Finds the vector equation of the plane with the both direction vectors and one coordinate (allow a sign slip), sets equal to the other coordinate, forms simultaneous equations and attempts to solve to find a value for $t = \ldots$ or $\lambda = \ldots$.

A1: Shows that the other coordinate lies on the plane by checking the third equation.

A1: Achieves the correct values $t = -2$ and $\lambda = 1$ or $t = 2$ and $\lambda = -1$ and concludes that the second coordinate lie on the plane; therefore, the lines lie on the same plane

Way 3

M1: Substitutes line 2 into line 1 and solves a pair of equations to find a value for t. Allow slip with the position of 0 and sign slips as long as the intention is clear.

A1: Solve two pairs of equations to achieve $t = 2$ for each.

A1: Achieves the correct value $t = 2$ and concludes that either

- a common point,
- the lines intersect
- the equations are consistent

therefore, the lines lie in the same plane

Way 4 (Using Further Pure 2 knowledge)

M1: A complete method to finds a vector which is normal to both lines and attempts to finds the equation of the plane containing one line.

A1: Achieves the correct equation for the plane containing each line.

A1: Conclusion, planes are the same, therefore the lines lie in the same plane.

(b) **This may be seen in part (a)**

B1: Correct **vector** equation allow any letter for the scalers.

Must start with $r = \ldots$ and uses two out of the following direction vectors $\pm\begin{pmatrix}1\\-1\\2\end{pmatrix}$, $\pm\begin{pmatrix}2\\-1\\3\end{pmatrix}$ or $\pm\begin{pmatrix}0\\-1\\1\end{pmatrix}$ and one of the following position vectors $\begin{pmatrix}1\\0\\3\end{pmatrix}$, $\begin{pmatrix}1\\-1\\4\end{pmatrix}$ or $\begin{pmatrix}3\\-2\\7\end{pmatrix}$

(c)

<u>Way 1</u>

M1: Calculates the scalar product between the direction vectors, allow one slip, if the intention is clear

dM1: Dependent on the previous method mark. Applies the scalar product formula with their scalar product to find a value for $\cos\theta$

A1: Correct answer only

<u>Way 2</u> (Using Further Pure 2 knowledge)

M1: Calculates the vector product between the direction vectors, allow one slip, if the intention is clear

dM1: Dependent on the previous method mark. Applies the vector product formula with their vector product to find a value for $\sin\theta$

A1: Correct answer only

Student Response A

(5)

$C_1 = \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-4}{3}$

$= \begin{pmatrix}1\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$

$C_2 = \begin{pmatrix}1\\0\\3\end{pmatrix} + t\begin{pmatrix}1\\-1\\2\end{pmatrix}$

~~on the same plane: multiple ways to join them with have same direction vector.~~

$C_1: \; 1+2\lambda \qquad C_2: 1+t$

$-1 - \lambda \qquad 0-t$

$4 + 3\lambda \qquad 3+2t$

$1+2\lambda = 1+t \qquad 2\lambda = t$

$-1-\lambda = 0-t \qquad -1-\lambda = -t \qquad t = \lambda+1$

$4+3\lambda = 3+2t \qquad 1+3\lambda = 2t$

$\lambda = 1.$

$t = 2.$

$\therefore$ they meet at a point and produce a plane.

b) : $\Pi = \begin{pmatrix}1\\0\\3\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\3\end{pmatrix} + \beta\begin{pmatrix}1\\-1\\2\end{pmatrix}$

c) $a \cdot b = |a||b|\cos\theta$

$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$

$\begin{pmatrix}2\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}1\\-1\\2\end{pmatrix} = 2 + 1 + 6$

$= 9$

$|a| = \sqrt{14}$ $\quad\therefore\quad 9 = \sqrt{14}\sqrt{6}\cos\theta$

$|b| = \sqrt{6}$ $\quad \frac{3\sqrt{21}}{14} = \cos\theta$

$\theta = \cos^{-1}\left(\frac{3\sqrt{21}}{14}\right)$

$= 79°$

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Examiner Comments

In part (a) Way 1

M1: Attempts to find the point of intersection of the two lines

A0: They do not check the third equation to show that the lines intersect

A0: The previous Accuracy mark needs to be scored before this mark is available.

In part (b)

B0: The equation of the plane does not start with *r* =

In part (c)

M1: Calculates the dot product of the direction vectors

M1: Applies the scalar product to find a value for cosθ

A0: Incorrect answer

Student Response B

(3)

$L_1 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$L_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$\lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \neq t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

for any value of λ or t so they are not parrallel.

$\begin{matrix} 1 \\ -1 \\ 4 \end{matrix} + \begin{matrix} 2\lambda \\ -\lambda \\ 3\lambda \end{matrix} = \begin{matrix} 1 \\ 0 \\ 3 \end{matrix} \begin{matrix} +t \\ -t \\ +2t. \end{matrix}$ ① ② ③

$2\lambda - t = 0.$ ①

$-\lambda + t = 1.$ ②

$\lambda = 1$

$t = 2.$

~~Equation t~~ $4 + 3(1) = 3 + 2(2)$

Equation is consistent for equation 3, meaning they intersect. Hence they are not skew, and as not parrallel they lie in the same plane.

(b). $\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(c). $\underset{\sim}{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ $\underset{\sim}{b} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$a \cdot b = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$= 2 - 1(-1) + 3(2)$

$= 9.$

$|a| = \sqrt{14}$

$|b| = \sqrt{6}.$

$\theta = \cos^{-1}\left(\frac{9}{\sqrt{14} \times \sqrt{6}} \right)$

$= 10.9^\circ$

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Examiner Comments

In part (a) Way 1

M1: Attempts to find the point of intersection of the two lines

A1: They check the third equation to show that the lines intersect

A1: They have concluded that the lines intersect concludes that therefore the lines lie on the same plane.

In part (b)

B0: The equation of the plane does not start with *r* =

In part (c)

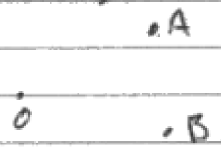
M1: Calculates the dot product of the direction vectors

M1: Applies the scalar product to find a value for cosθ

A0: Correct angle but is not given to the nearest degree which was the demand of the question.

Student Response C

a.) $l_1: r = \begin{pmatrix}1\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$



$l_2: r = \begin{pmatrix}1\\0\\3\end{pmatrix} + t\begin{pmatrix}1\\-1\\2\end{pmatrix}$

$\begin{pmatrix}1+2\lambda\\-1-\lambda\\4+3\lambda\end{pmatrix} = \begin{pmatrix}1+t\\-t\\3+2t\end{pmatrix}$

$2\lambda - t = 0 \quad (1)$

$-\lambda + t = 1 \quad (2)$

$3\lambda - 2t = -1 \quad (3)$

Solve using 1 and 2

$\lambda = 1 \quad t = 2$

$3 \times 1 - 2 \times 2 = -1$ ← sub into 3

λ and t satisfy all equations so lines intersect

b) $r = \begin{pmatrix}1\\-1\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\3\end{pmatrix} + t\begin{pmatrix}0\\1\\-1\end{pmatrix}$

$\vec{AB} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$

c) $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 2 + 1 + 6$
$= 9$

$\cos\theta = \dfrac{9}{\sqrt{2^2+1+3^2} \times \sqrt{1+1+2^2}}$

$= \dfrac{9}{\sqrt{14} \times \sqrt{6}}$

$\cos\theta = \dfrac{3\sqrt{21}}{14}$

$\theta = 10.89°$

$= 11°$

6/7

Examiner Comments

In part (a) Way 1

M1: Attempts to find the point of intersection of the two lines

A1: They check the third equation to show that the lines intersect

A0: They have concluded that the lines intersect but does not conclude that therefore the lines lie on the same plane, the demand of the question.

In part (b)

B1: A correct equation of the plane which starts with *r* =

In part (c)

M1: Calculates the dot product of the direction vectors

M1: Applies the scalar product to find a value for cosθ

A1: Correct angle to the nearest degree

Exemplar Question 8

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8. A scientist is studying the effect of introducing a population of white-clawed crayfish into a population of signal crayfish.
At time t years, the number of white-clawed crayfish, w, and the number of signal crayfish, s, are modelled by the differential equations

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{5}{2}(w - s)$$
$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{2}{5}w - 90\mathrm{e}^{-t}$$

(a) Show that

$$2\frac{\mathrm{d}^2w}{\mathrm{d}t^2} - 5\frac{\mathrm{d}w}{\mathrm{d}t} + 2w = 450\mathrm{e}^{-t}$$

(3)

(b) Find a general solution for the number of white-clawed crayfish at time t years.

(6)

(c) Find a general solution for the number of signal crayfish at time t years.

(2)

The model predicts that, at time T years, the population of white-clawed crayfish will have died out.

Given that $w = 65$ and $s = 85$ when $t = 0$

(d) find the value of T, giving your answer to 3 decimal places.

(6)

(e) Suggest a limitation of the model.

(1)

(Total for Question 8 is 18 marks)

Mean Score 11.6 out of 18

Examiner Comments

This question is assessing a pair of coupled first order differential equations and solving second order differential equations (spec ref 9.5, 9.6, 9.9)

In part (a), the majority of candidates answered this part well and the most common errors were sign slips or numerical copying mistakes. Some candidates used dot notation despite the question been written with $\frac{dw}{dt}$ and failed to give the answer as required.

For part (b) Most answered this well with most finding the Auxiliary Equation correctly. The most common error occurred with the Particular Integral with an incorrect format, such as λte^{-t}, or poor differentiation. A small minority mistakenly used *x* instead of *t*.

A significant number of students did not attempt part (c) of the question. Many successfully gained the method mark, but there were frequent sign errors in applying "$-\frac{2}{5}\ \frac{dw}{dt}$". A significant minority replaced their *w* in the second equation and integrated to find *s*. Only one solution that I saw attempted to find a constant for their solution.

In part (d) the majority of students gained the first two marks in this section but failed to proceed beyond setting $w = 0$. A significant number failed to create a 3TQ in $e^{1.5t}$ because they didn't multiply the whole equation by $e^{-t}$and had little chance of gaining the last 4 marks. There was mixed success in solving the 3TQ – some made a substitution using $x = e^{1.5\,t}$, but others attempted more complicated substitutions, or tried to square their equation with little progress. Those who solved their 3TQ were able to correctly undo the logs, but failed to round correctly.

Part (e) was very poorly answered with most students talking about other factors instead of the potential for negative values, they did not take the hint from part (d) finding the time when $w = 0$. Comments were rarely in context and too many failed to mention the validity of the model.

Mark Scheme

Question	Scheme	Marks	AOs
8(a)	$\frac{d^2w}{dt^2}=\frac{5}{2}\left(\frac{dw}{dt}-\frac{ds}{dt}\right)$ or $\frac{ds}{dt}=\frac{dw}{dt}-\frac{2}{5}\frac{d^2w}{dt^2}$ o.e.	B1	1.1b
	$\frac{ds}{dt}=\frac{dw}{dt}-\frac{2}{5}\frac{d^2w}{dt^2}\Rightarrow\frac{dw}{dt}-\frac{2}{5}\frac{d^2w}{dt^2}=\frac{2}{5}w-90e^{-t}$	M1	2.1
	$2\frac{d^2w}{dt^2}-5\frac{dw}{dt}+2w=450e^{-t}$ *	A1*	1.1b
		(3)	
(b)	$2m^2-5m+2=0\Rightarrow m=...$	M1	3.4
	$m=2,\frac{1}{2}$	A1	1.1b
	$(w)=Ae^{\alpha t}+Be^{\beta t}$	M1	3.4
	$(w)=Ae^{0.5t}+Be^{2t}$	A1	1.1b
	PI: Try $w=ke^{-t}\Rightarrow\frac{dw}{dt}=-ke^{-t}\Rightarrow\frac{d^2w}{dt^2}=ke^{-t}$ $2ke^{-t}+5ke^{-t}+2ke^{-t}=450e^{-t}\Rightarrow k=...$	M1	3.4
	$w=$ 'their C.F.' $+\ 50e^{-t}$ $\left(w=Ae^{0.5t}+Be^{2t}+50e^{-t}\right)$	A1ft	1.1b
		(6)	
(c)	$s=w-\frac{2}{5}\frac{dw}{dt}=Ae^{0.5t}+Be^{2t}+50e^{-t}-\frac{2}{5}\left(\frac{A}{2}e^{0.5t}+2Be^{2t}-50e^{-t}\right)$	M1	3.4
	$s=\frac{4A}{5}e^{0.5t}+\frac{B}{5}e^{2t}+70e^{-t}$	A1	1.1b
		(2)	
(d)	$65=A+B+50,\ 85=\frac{4A}{5}+\frac{B}{5}+70\Rightarrow A=...,B=...$ $(\text{NB } A=20\ \ B=-5)$	M1	3.3
	$w=0\Rightarrow 20e^{0.5t}-5e^{2t}+50e^{-t}=0$	dM1	1.1b
	$e^{3t}-4e^{1.5t}-10(=0)$ or a multiple	A1	3.1a
	$e^{1.5t}=\frac{4\pm\sqrt{4^2-4\times(1)(-10)}}{2}$	M1	1.1b
	$1.5t=\ln\left(\frac{4+\sqrt{56}}{2}\right)$	M1	2.3
	$T=\frac{2}{3}\ln\left(\frac{4+\sqrt{56}}{2}\right)=$ awrt 1.165	A1	3.2a
		(6)	
(e)	E.g. • Either population becomes negative which is not possible	B1	3.5b

	• When the white-clawed crayfish have died out, the model will not be valid		
		(1)	
			(18 marks)

Notes

(a)
B1: Differentiates the first equation with respect to t correctly.
M1: Substitutes $\frac{\mathrm{d}s}{\mathrm{d}t}$ into their derivative.
A1*: Achieves the printed answer with no errors.

(b) **<u>Note:</u> All the mark except the final A1 are available if they use other variables.**
M1: Uses the model to form and solve the Auxiliary Equation.
A1: Correct roots of the AE.
M1: Uses the model to form the Complementary Function for their roots (they may be complex roots)
A1: Correct CF
M1: Chooses the correct form of the PI according to the model and uses a complete method to find the PI. Uses $w = ke^{-t}$ finds both $\frac{\mathrm{d}w}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2w}{\mathrm{d}t^2}$ substitutes into the differential equation and find the value of k.
A1ft: Dependent on all three of the previous method marks. Following through on their CF only to give $w = \text{'their CF'} + 50\mathrm{e}^{-t}$

(c)
M1: Substitutes into the first equation the answer for part (b) in place of w and the derivative of their (b) in place of $\frac{\mathrm{d}w}{\mathrm{d}t}$. If they rearrange to make S the subject first and make a slip but still substitutes for w and $\frac{\mathrm{d}w}{\mathrm{d}t}$ allow this mark.
A1: Correct simplified equation.

(d)
M1: Uses the initial conditions $t = 0$, $w = 65$ and $s = 85$ to form simulations equations and solves to find the values of their constants
dM1: Dependent on the previous method mark. Sets $w = 0$
A1: Processes the indices correctly to obtain a 3-term quadratic equation in terms of $\mathrm{e}^{1.5t}$. It does not need to all be on one side and condone missing = 0.
M1: Solves their three-term quadratic (3TQ) to reach $\mathrm{e}^{pt} = q$ where the value of p must be consistent with their quadratic of the form $\mathrm{A}\mathrm{e}^{2pt} + \mathrm{B}\mathrm{e}^{pt} + \mathrm{C} = 0$
M1: Correct use of logarithms to reach $pt = \ln q$ where $q > 0$ and rejects the other solution
A1: awrt 1.165
Note: the final 3 marks only can be implied by a correct answer following the correct 3-term quadratic equation in terms of $\mathrm{e}^{1.5t}$

(e)
B1: Suggests a suitable limitation of the model, not valid when negative population
Any mention of other factors such as does not take into account e.g. other predictors, fishing, disease, lack of food etc is B0

Student Response A

$$\frac{dw}{dt} = \frac{5}{2}w - \frac{5}{2}s$$

b) aux eq.

$2m^2 - 5m + 2 = 0$

$(m-2)(2m-1)$

$m = 2 \qquad m = 1/2$

General

$Ae^{2t} + Be^{1/2 t}$

$w = ae^{-t}$

$\frac{dw}{dt} = -ae^{-t}$

$\frac{d^2w}{dt^2} = ae^{-t}$

$2(ae^{-t}) - 5(-ae^{-t}) + 2(ae^{-t}) = 450e^{-t}$

$2a + 5a + 2a = 450$

$9a = 450$

$a = 50$

General $= Ae^{2t} + Be^{1/2 t} + 50e^{-t} = 0$

5/18

Examiner Comments

In part (a)
B0 M0 A0: No attempt

In part (b)
M1: Finds the auxiliary equation and solve to find a value for m
A1: Correct values for m
M1 A1: Forms the correct Complementary function
M1: Uses the correct particular integral form and finds the value of the constant
A0: Does not have an equation starting $w = \ldots$

In parts (c), (d) and (e)
There is no attempt

Student Response B

a) $\frac{dw}{dt} = \frac{5}{2}w - \frac{5}{2}s$

$\frac{5}{2}w - \frac{dw}{dt} = \frac{5}{2}s$

$s = w - \frac{2}{5}\frac{dw}{dt}$ ①

$\frac{ds}{dt} = \frac{dw}{dt} - \frac{2}{5}\frac{d^2w}{dt^2}$

Sub into ②: $\frac{dw}{dt} - \frac{2}{5}\frac{d^2w}{dt^2} = \frac{2}{5}w - 90e^{-t}$

$\frac{2}{5}\frac{d^2w}{dt^2} - \frac{dw}{dt} + \frac{2}{5}w = 90e^{-t}$

$2\frac{d^2w}{dt^2} - 5\frac{dw}{dt} + 2w = 450e^{-t}$

b) auxiliary equation: $2m^2 - 5m + 2 = 0$

$(m-2)(2m-1) = 0$

$m = 2$ or $m = \frac{1}{2}$

complementary function in form: $Ae^{2t} + Be^{\frac{1}{2}t}$

particular integral in form λe^{-t}

let $w = \lambda e^{-t}$

$\frac{dw}{dt} = -\lambda e^{-t}$

$\frac{d^2w}{dt^2} = \lambda e^{-t}$

Sub into second order differential equation:

$2\lambda e^{-t} + 5\lambda e^{-t} + 2\lambda e^{-t} = 450e^{-t}$

$9\lambda e^{-t} = 450e^{-t}$

$\lambda = 50$

general solution: $w = Ae^{2t} + Be^{\frac{1}{2}t} + 50e^{-t}$

c) $\frac{dw}{dt} = 2Ae^{2t} + \frac{1}{2}Be^{\frac{1}{2}t} - 50e^{-t}$

Sub into equation ①:

$S = Ae^{2t} + Be^{\frac{1}{2}t} + 50e^{-t} - \frac{4}{5}Ae^{2t} - \frac{1}{5}Be^{\frac{1}{2}t} - 20e^{-t}$

$S = \frac{A}{5}e^{2t} + \frac{4B}{5}e^{\frac{1}{2}t} + 30e^{-t}$

d) when $t = 0$ $w = 65$

$65 = Ae^{0} + Be^{0} + 50e^{0}$

$15 = A + B$ $\boxed{15 - B = A}$ ③

when $t = 0$ $S = 85$

$85 = \frac{A}{5}e^{0} + \frac{4B}{5}e^{0} + 30e^{0}$

$55 = \frac{A}{5} + 4\frac{B}{5}$ $275 = A + 4B$

Sub in ③: $275 = 15 - B + 4B$

$260 = 3B$ $B = \frac{260}{3}$

$15 = A + \frac{260}{3} \quad A = -\frac{215}{3}$

$W = -\frac{215}{3}e^{2t} + \frac{260}{3}e^{\frac{1}{2}t} + 50e^{-t}$

when $t \neq 0$ $W = 0$

$0 = \frac{260}{3}e^{\frac{1}{2}t} + 50e^{-t} - \frac{215}{3}e^{2t}$

$\frac{215}{3}e^{2t} = \frac{260}{3}e^{\frac{1}{2}t} + 50e^{-t}$

$\ln\frac{215}{3} + 2t = \ln\frac{260}{3} + \frac{1}{2}t + \ln 50 - t$

$\frac{5}{2}t = \ln\frac{260}{3} + \ln 50 - \ln\frac{215}{3}$

$t = \frac{2}{5}\left(\ln\frac{260}{3} + \ln 50 - \ln\frac{215}{3}\right)$

$= 1.641$ years

e) The numbers of each species is likely to be affected by other factors that aren't included such as availability of food sources and competition.

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Examiner Comments

In part (a)
B1: Correct differentiation of the first equation
M1: Substitutes $\frac{ds}{dt}$ into the second equation
A1: Achieves the printed answer with no errors.

In part (b)
M1: Finds the auxiliary equation and solve to find a value for m
A1: Correct values for m
M1 A1: Forms the correct Complementary function
M1: The correct form for the Particular Integral and uses the correct method to find the constant.
A1: Correct general equation

In part (c)
M1: Substitutes into the first equation the answer for part (b) in place of w and the derivative of their (a) in place of $\frac{dw}{dt}$, there is a sign slip
A0: Incorrect equation.

In part (d)
M1: Uses $t = 0$, $w = 65$ and $s = 85$ to form simulations equations and solves to find the values of their constants
M1: Sets $w = 0$
The key step to solve this question is to realise that it is a quadratic equation for $e^{1.5t}$
A0: Does not have the correct three term quadratic for $e^{1.5t}$
M0: They do not have a three-term quadratic for $e^{1.5t}$ so it cannot be solved and this mark cannot be awarded.
M0: Does not use logarithms to solve $e^{pt} = q$ to reach $pt = \ln q$
A0: Follows previous M0

In part (e)
B0: They comment on external factors not the model, they do not make the connection with part (d), finding when the population of white-clawed crayfish die out, the population cannot be negative.

Student Response C

$$\frac{d^2w}{dt^2} = \frac{5}{2}\frac{dw}{dt} - 2.5\frac{ds}{dt}$$

$$\frac{d^2w}{dt^2} = 2.5\frac{dw}{dt} - 2.5\left(0.4w - 90e^{-t}\right)$$

$$\frac{d^2w}{dt^2} - 2.5\frac{dw}{dt} + w = 225e^{-t}$$

$$2\frac{d^2w}{dt^2} - 5\frac{dw}{dt} + 2w = 450e^{-t}$$

$$2m^2 - 5m + 2 = 0$$

$$(m-2)(2m-1)$$

$$m = 2, \tfrac{1}{2}$$

$$w = Ae^{2t} + Be^{\frac{t}{2}}$$

$$w = \lambda e^{-t}$$

$$\frac{dw}{dt} = -\lambda e^{-t}$$

$$\frac{d^2w}{dt^2} = \lambda e^{-t}$$

$$2\lambda e^{-t} + 5\lambda e^{-t} + 2\lambda e^{-t} = 450e^{-t}$$

$$9\lambda = 450$$

$$\lambda = 50$$

$$w = Ae^{2t} + Be^{\frac{t}{2}} + 50e^{-t}$$

$$\frac{dw}{dt} = 2Ae^{2t} + \tfrac{1}{2}Be^{\frac{t}{2}} - 50e^{-t}$$

$$s = -0.4\frac{dw}{dt} + w$$

$$-0.4\frac{dw}{dt} = -0.8Ae^{2t} - 0.2Be^{\frac{t}{2}} + 20e^{-t}$$

$$w - 0.4\frac{dw}{dt} = 0.2Ae^{2t} + 0.8Be^{\frac{t}{2}} + 70e^{-t}$$

$$s = 0.2Ae^{2t} + 0.8Be^{\frac{t}{2}} + 70e^{-t}$$

$w = 65 \quad s = 85 \quad t = 0$

$65 = A + B + 50$

$85 = 0.2A + 0.8B + 70$

$A + B = 15$

$0.2A + 0.8B = 15$

$A = -5 \qquad B = 20$

$w = -5e^{2t} + 20e^{\frac{1}{2}t} + 50e^{-t}$

$w = 0$

$0 = -5e^{2t} + 20e^{\frac{1}{2}t} + 50e^{-t}$

$0 = 5(4e^{\frac{1}{2}T} - e^{2T} + 10e^{-T})$

$5(4e^{\frac{3T}{2}} - e^{3T} + 10)$

~~$\frac{-4 \pm \sqrt{(4)^2 - (4 \times -1 \times 10)}}{2 \times -1}$~~

~~$\frac{-4 \pm \sqrt{56}}{-2}$~~

$e^{3t} - 4e^{\frac{3T}{2}} - 10 = 0$

$e^{\frac{3T}{2}} = 2 \pm \sqrt{14}$

$\frac{3T}{2} = \ln|2 \pm \sqrt{14}|$

$T = \frac{2}{3}\ln|2 \pm \sqrt{14}|$

$T = \frac{2}{3}\ln|2 + \sqrt{14}|$

$T = 1.165$

A limitation of the model is that it does not account for external factors e.g. An increase food supply may increase the number of crayfish

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Examiner Comments

In part (a)

B1: Correct differentiation of the first equation

M1: Substitutes $\frac{ds}{dt}$ into the second equation

A1: Achieves the printed answer with no errors.

In part (b)

M1: Finds the auxiliary equation and solve to find a value for *m*

A1: Correct values for *m*

M1 A1: Forms the correct Complementary function

M1: The correct form for the Particular Integral and uses the correct method to find the constant.

A1: Correct general equation

In part (c)

M1: Substitutes into the first equation the answer for part (b) in place of *w* and the derivative of their (a) in place of $\frac{dw}{dt}$.

A1: Correct equation of *s*.

In part (d)

M1: Uses $t = 0$, $w = 65$ and $s = 85$ to form simulations equations and solves to find the values of their constants

M1: Sets $w = 0$

The key step to solve this question is to realise that it is a quadratic equation for $e^{1.5t}$

A1: Forms a correct three term quadratic for $e^{1.5t}$

M1: Solves their three-term quadratic for $e^{1.5t}$

M1: Uses logarithms to solve $e^{pt} = q$ to reach $pt = \ln q$

A1: Correct value for *t*.

In part (e)

B0: They comment on external factors not the model, they do not make the connection with part (d), finding when the population of white-clawed crayfish die out, the population cannot be negative.

A Level Further Mathematics – Core Pure 2 (9FM0 02)

Exemplar Question 1

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1 (*a*) Prove that

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -k < x < k$$

stating the value of the constant *k*.

(5)

(*b*) Hence, or otherwise, solve the equation

$$2x = \tanh\left(\ln\sqrt{2 - 3x}\right)$$

(5)

(Total for Question 1 is 10 marks)

Mean Score 6.9 out of 10

Examiner Comments

Part (a) required proving the logarithmic form of $\tanh^{-1}(x)$ and it produced a mixed response. Most candidates were able to obtain an appropriate equation in exponentials but some did not appreciate the need to rearrange it, particularly those who did not introduce another variable. Those who did make $e^{2"y"}$ the subject usually did so correctly, although a few did not convert the resulting $\frac{-1-x}{x-1}$ into the required $\frac{1+x}{1-x}$. Some solved a quadratic in e^y but often got bogged down in the algebra. A small number of students started with the given result and verified it appropriately. A few attempted to use $\tanh^{-1} x = \frac{\sinh^{-1} x}{cosh^{-1} x}$. A significant number neglected to state the value of *k*.

Part (b) was more successful for most although slips occasionally led to students not achieving a quadratic when the logarithms were removed. There were a surprising number of errors seen producing the correct quadratic equation from $\frac{1+2x}{1-2x} = 2 - 3x$. The quadratic was almost always solved correctly, but many failed to reject the ineligible solution, despite being asked about the range of validity of $\tanh^{-1}(x)$ in part (a).
Presentation of work was an issue for many. For example, many scripts were seen where "tan" was written when "tanh" was intended.

Mark Scheme

Question	Scheme	Marks	AOs
1(a)	$y=\tanh^{-1}(x)\Rightarrow \tanh y = x \Rightarrow x=\dfrac{\sinh y}{\cosh y}=\dfrac{e^y-e^{-y}}{e^y+e^{-y}}$	M1 A1	2.1 1.1b
	Note that some candidates only have one variable and reach e.g. $x=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ or $\tanh x=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$ Allow this to score M1A1		
	$x(e^{2y}+1)=e^{2y}-1\Rightarrow e^{2y}(1-x)=1+x\Rightarrow e^{2y}=\dfrac{1+x}{1-x}$	M1	1.1b
	$e^{2y}=\dfrac{1+x}{1-x}\Rightarrow 2y=\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow y=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)*$	A1*	2.1
	Note that $e^{2y}(x-1)+x+1=0$ can be solved as a quadratic in e^y: $e^y=\dfrac{-\sqrt{0-4(x-1)(x+1)}}{2(x-1)}=\dfrac{-\sqrt{4(1-x)(x+1)}}{2(x-1)}=\dfrac{2\sqrt{(1-x)(x+1)}}{2(1-x)}$ $=\dfrac{\sqrt{(x+1)}}{\sqrt{(1-x)}}\Rightarrow y=\dfrac{1}{2}\ln\dfrac{(x+1)}{(1-x)}*$ Score **M1** for an attempt at the quadratic formula to make e^y the subject (condone ± √…) and **A1*** for a correct solution that rejects the positive root at some point and deals with the $(x – 1)$ bracket correctly		
	$k = 1$ or $-1 < x < 1$	B1	1.1b
		(5)	
(a) Way 2	$\tanh^{-1}x=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)\Rightarrow x=\tanh\left(\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)\right)=\dfrac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1}$	M1 A1	2.1 1.1b
	$x=\dfrac{e^{\ln\frac{1+x}{1-x}}-1}{e^{\ln\frac{1+x}{1-x}}+1}=\dfrac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1}=x$ Hence true, QED, tick etc.	M1 A1	1.1b 2.1
(b)	$2x=\tanh\left(\ln\sqrt{2-3x}\right)\Rightarrow \tanh^{-1}(2x)=\ln\sqrt{2-3x}$	M1	3.1a
	$\dfrac{1}{2}\ln\left(\dfrac{1+2x}{1-2x}\right)=\dfrac{1}{2}\ln(2-3x)\Rightarrow\dfrac{1+2x}{1-2x}=2-3x$	M1	2.1
	$6x^2-9x+1=0$	A1	1.1b
	$6x^2-9x+1=0\Rightarrow x=\ldots$	M1	1.1b
	$x=\dfrac{9-\sqrt{57}}{12}$	A1	3.2a
		(5)	

	Alternative for first 2 marks of (b)		
	$2x = \tanh\left(\ln\sqrt{2-3x}\right) \Rightarrow 2x = \dfrac{e^{2\ln\sqrt{2-3x}}-1}{e^{2\ln\sqrt{2-3x}}+1}$	M1	3.1a
	$\Rightarrow \dfrac{2-3x-1}{2-3x+1} = 2x$	M1	2.1
			(10 marks)

Notes

(a)

<u>If you come across any attempts to use calculus to prove the result – send to review</u>

M1: Begins the proof by expressing tanh in terms of exponentials and forms an equation in exponentials.

The exponential form can be any of $\dfrac{\left(e^{y}-e^{-y}\right)/2}{\left(e^{y}+e^{-y}\right)/2}, \dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}}, \dfrac{e^{2y}-1}{e^{2y}+1}$

Allow any variables to be used **but the final answer must be in terms of *x***. Allow alternative notation for $\tanh^{-1}x$ e.g. artanh, arctanh.

A1: Correct expression for "x" in terms of exponentials

M1: Full method to make $e^{2"y"}$ the subject of the formula. This must be correct algebra so allow sign errors only.

A1*: Completes the proof by using logs correctly and reaches the printed answer with no errors.

Allow e.g. $\frac{1}{2}\ln\left(\frac{x+1}{1-x}\right), \frac{1}{2}\ln\frac{x+1}{1-x}, \frac{1}{2}\ln\left|\frac{x+1}{1-x}\right|$. Need to see $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ as a conclusion

but allow if the proof concludes that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ with y defined as $\tanh^{-1}x$ earlier.

B1: Correct value for k or writes $-1 < x < 1$

Way 2

M1: Starts with result, takes tanh of both sides and expresses in terms of exponentials

A1: Correct expression

M1: Eliminates exponentials and logs and simplifies

A1: Correct result (i.e. $x = x$) with conclusion

B1: Correct value for k or writes $-1 < x < 1$

(b)

M1: Adopts a correct strategy by taking $\tanh^{-1}$ of both sides

M1: Makes the link with part (a) by replacing artanh($2x$) with $\frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right)$ and demonstrates the use of the power law of logs to obtain an equation with logs removed **correctly**.

A1: Obtains the correct 3TQ

M1: Solves their 3TQ using a correct method (see General Guidance – if no working is shown (calculator) and the roots are correct for their quadratic, allow M1)

A1: Correct value with the other solution rejected (accept rejection by omission) so $x = \dfrac{9\pm\sqrt{57}}{12}$ scores A0 unless the positive root is rejected

Alternative for first 2 marks of (b)

M1: Adopts a correct strategy by expressing tanh in terms of exponentials

M1: Demonstrates the use of the power law of logs to obtain an equation with logs removed correctly

Student Response A

a) $\tanh x = \frac{e^x - e^{-x}}{2} \div \frac{e^x + e^{-x}}{2}$

$= \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$

$\text{arctanh}\, x = \frac{e^{2x} + 1}{e^{2x} - 1}$

$\ln\left(\frac{e^{2x}+1}{e^{2x}-1}\right)$

$\ln(e^{2x}+1) - \ln(e^{2x}-1)$.

b) $x = \tanh \frac{1}{2} \ln \frac{(1+x)}{(1-x)}$

2/10

Examiner Comments: (a) M1A1M0A0B0 (b) M0M0A0M0A0

In part (a), this candidate makes an appropriate start by expressing tanh *x* in terms of exponentials correctly, thus scoring the first 2 marks. However, no further progress is made as there is no attempt to rearrange, possibly because another variable is not introduced.

In part (b) the candidate does not appear to be able to make the connection with part (a) and makes no progress with the solution.

Student Response B

let ~~...~~ $\tanh^{-1} x = y$

~~$y = \tanh x$~~

$\tanh y = x$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$

$= \frac{e^{2y} - 1}{e^{2y} + 1}$

$xe^{2y} + x - e^{2y} + 1 = 0$

$e^{2y}(x - 1) + x + 1 = 0$

$e^{2y} = \frac{-x - 1}{x - 1} = \frac{1 + x}{1 - x}$

$2y = \ln\left|\frac{1+x}{1-x}\right|$

$y = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$

$y = \tanh^{-1}(x) = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$

b) $\tanh^{-1}(2x) = \frac{1}{2}\ln\left(\frac{1+2x}{1-2x}\right) = \ln\sqrt{2-3x}$

$\ln\left(\left(\frac{1+2x}{1-2x}\right)^2\right) = \ln(2-3x)^{1/2}$

$\frac{1 + 4x + 4x^2}{1 - 4x + 4x^2} = (2-3x)^{1/2}$

~~$1 + 8x + 24x^2 + 8x^3$~~

$(1+2x)^4 = 1 + 8x + 24x^2 + 32x^3 + 16x^4$

$(1-2x)^4 = 1 - 8x + 24x^2 - 32x^3 + 16x^4$

$1 + 8x + 24x^2 + 32x^3 + 16x^4 = (2-3x)(1 - 8x + 24x^2 - 32x^3 + 16x^4)$

$= 2 - 16x + 48x^2 - 64x^3 + 32x^4 - 3x + 24x^2 - 72x^3 + 96x^4 - 48x^5$

$1 + 4x + 4x^2 = (\sqrt{2-3x})(1 - 4x + 4x^2)$

$1 + 4x + 4x^2 = \sqrt{2-3x} - 4\sqrt{2-3x} + 4x^2\sqrt{2-3x}$

$u = \sqrt{2-3x} \qquad \frac{2-u^2}{3} = x$

$1 + \frac{8-4u^2}{3} + \frac{4-4u^2+u^4}{9} = u - 4u + \frac{4u-4u^3+u^5}{9}$

$u^5 - u^4 - 4u^3 + 16u^2 - 32u - 37 = 0$

$u =$

5/10

Examiner Comments: (a) M1A1M1A1B0 (b) M1M0A0M0A0

In part (a), this candidate shows the required result correctly. Note that the final result is an acceptable conclusion as y has been defined as artanh x previously. As with many candidates, there is no reference to the required value of k for the domain and so the B mark was not scored.

In part (b) the candidate makes the connection with part (a) but the logs are not removed correctly and this results in the candidate not obtaining a quadratic equation which means that no further marks are scored after the first method mark.

Student Response C

a) $\tanh x = \frac{e^{x}-e^{-x}}{2} \div \frac{e^{x}+e^{-x}}{2}$

$= \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

$\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$

$\text{arctanh}\, x = \frac{e^{2x}+1}{e^{2x}-1}$

$\ln\left(\frac{e^{2x}+1}{e^{2x}-1}\right)$

$\ln(e^{2x}+1) - \ln(e^{2x}-1).$

~~[illegible]~~

b) $x = \tanh \frac{1}{2} \ln \frac{(1+x)}{(1-x)}$

10/10

Examiner Comments: (a) M1A1M1A1B1 (b) M1M1A1M1A1

In part (a), this candidate shows the required result correctly. Note that this candidate takes an unusual approach when making *y* the subject (by solving their equation using the quadratic formula). The candidate also states the correct value for *k*.

In part (b) the candidate has a correct solution and importantly at the end, selects the relevant solution.

Exemplar Question 2

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2 The roots of the equation

$$x^3 - 2x^2 + 4x - 5 = 0$$

are p, q and r.

Without solving the equation, find the value of

(i) $\frac{2}{p} + \frac{2}{q} + \frac{2}{r}$

(ii) $(p - 4)(q - 4)(r - 4)$

(iii) $p^3 + q^3 + r^3$

(8)

(Total for Question 2 is 8 marks)

Mean Score 5.8 out of 8

Examiner Comments

Question 2 involved the roots of a cubic equation and it was common to be awarding the first six marks for parts (i) and (ii). Most began their answer with the correct values for the sum, pair sum and product with sign errors being very rare. In part (i) the vast majority expressed the new sum correctly in terms of the pair sum and product and proceeded correctly. A small number thought that $\frac{2}{p} + \frac{2}{q} + \frac{2}{r}$ was equal to $\frac{2(p+q+r)}{pqr}$. The alternative of substituting $x = \frac{2}{y}$ to find the cubic in y was not common but was usually correct.

In part (ii), most were able to multiply out and obtain an expression that could be evaluated. A common error was the absence of the constant – 64. As with (i), the alternative was not widely seen.

Part (iii) proved very difficult since only a relatively small number of students were able to recall a correct identity for the sum of the cubes. Those who tried to produce one by expanding were almost always unsuccessful.

Mark Scheme

Question	Scheme	Marks	AOs
2(i)	$p+q+r=2,\quad pq+pr+qr=4,\quad pqr=5$	B1	3.1a
	$\frac{2}{p}+\frac{2}{q}+\frac{2}{r}=\frac{2(pq+pr+qr)}{pqr}$	M1	1.1b
	$=\frac{8}{5}$	A1ft	1.1b
		(3)	
	Alternative for part (i)		
	$x=\frac{2}{y}\Rightarrow\frac{8}{y^3}-\frac{8}{y^2}+\frac{8}{y}-5=0\Rightarrow 5y^3-8y^2+8y-8=0$	B1	3.1a
	$\frac{2}{p}+\frac{2}{q}+\frac{2}{r}=-\frac{-8}{5}$	M1	1.1b
	$=\frac{8}{5}$	A1ft	1.1b
		(3)	
(ii)	$(p-4)(q-4)(r-4)=(pq-4p-4q+16)(r-4)$ $=pqr-4pq-4pr-4qr+16p+16q+16r-64$	M1 A1	1.1b 1.1b
	$\left(=pqr-4(pq+pr+qr)+16(p+q+r)-64\right)$		
	$=5-4(4)+16(2)-64=-43$	A1	1.1b
		(3)	
	Alternative for part (ii)		
	$(x+4)^3-2(x+4)^2+4(x+4)-5=0$	M1	1.1b
	$=\ldots64+\ldots-32+\ldots16+\ldots-5=43$	A1	1.1b
	$\therefore(p-4)(q-4)(r-4)=-43$	A1	1.1b
		(3)	
(iii)	E.g. $p^3+q^3+r^3=$ $=(p+q+r)^3-3(p+q+r)(pq+pr+qr)+3pqr$ or $=(p+q+r)\left((p+q+r)^2-2(pq+pr+qr)-pq-pr-qr\right)+3pqr$ or $=2\left((p+q+r)^2-2(pq+pr+qr)\right)-4(p+q+r)+3pqr$ $\Rightarrow p^3+q^3+r^3=\ldots$	M1	3.1a
	$=2^3-3(2)(4)+3(5)=-1$ $=2\left(2^2-3(4)\right)+3(5)=-1$ $=2\left(2^2-2(4)\right)-4(2)+3(5)=-1$	A1	1.1b
		(2)	

	Alternative for part (iii)		
	$p^3-2p^2+4p-5=0,\ q^3-2q^2+4q-5=0,\ r^3-2r^2+4r-5=0$ $p^3+q^3+r^3-2\left(p^2+q^2+r^2\right)+4\left(p+q+r\right)-15=0$ $p^3+q^3+r^3=2\left(\left(p+q+r\right)^2-2\left(pq+pr+qr\right)\right)-4\left(p+q+r\right)+15$ $\Rightarrow p^3+q^3+r^3=...$	M1	3.1a
	$=2\left(2^2-2\left(4\right)\right)-4\left(2\right)+15=-1$	A1	1.1b
		(2)	

(8 marks)

Notes

(i)
B1: Identifies the correct values for all 3 expressions (can score anywhere). Allow notation such as $\sum p,\ \sum pq$ for the sum and pair sum.
M1: Uses a correct identity for the sum
A1ft: Correct value (follow through their 2, 4 and 5)
Alternative:
B1: Obtains the correct cubic in "y"
M1: Uses a correct method
A1ft: Correct value (follow through their 2, 4 and 5)
(ii)
M1: Attempt to expand – must have an expression that involves the sum, pair sum and product
A1: Correct expansion
A1: Correct value
Alternative:
M1: Substitutes $x + 4$ for x in the given cubic
A1: Calculates the correct constant term
A1: Correct value
(iii)
M1: Establishes a correct identity that is in terms of the sum, pair sum and product and substitutes to reach a numerical expression for $p^3+q^3+r^3$
A1: Correct value

Student Response A

i) $\frac{2}{p}+\frac{2}{q}+\frac{2}{r} = \frac{2qr+2pr+2pq}{pqr}$

$\sum\alpha = 2$

$\sum\alpha\beta = 4$ $\Rightarrow \frac{2\sum\alpha}{\sum\alpha\beta\gamma} = \frac{4}{5}$

$\sum\alpha\beta\gamma = 5$

ii) $(p-4)(q-4)(r-4) = \cancel{pqr - 4\sum\alpha\beta}$

$(pq - 4(p+q) + 16)(r-4) = pqr - 4\sum\alpha\beta + 16\sum\alpha$

$= 5 - 4(4) + 16(2) = 21$

iii) $p^3+q^3+r^3 = (p+q+r)^3 - 3pqr - 3\sum\alpha\beta \times \sum\alpha$

$= (2)^3 - 3(5) - 3(4)(2)$

$= 17$

$(p+q+r)^3 = 3pqr + p^3 + q^3 + r^3 + 3(p^2q + p^2r + pq^2 + pr^2 + \ldots) + \ldots$

3/8

Examiner Comments: (i) B1M1A0 (ii) M1A0A0 (iii) M0A0

In part (i), this candidate writes down the correct values for the sum, pair sum and product and then uses a correct identity for the sum of the reciprocals but makes a mistake with its evaluation.

In part (ii), this candidate makes progress in expanding the brackets but has omitted the "– 64" and so only gains the first mark in this part.

In part (iii), the candidate does use a correct identity and so scores no marks.

Student Response B

i) $\frac{2pq+2pr+2qr)}{pqr} = \frac{2(pq+pr+qr)}{pqr}$

$pq+pr+qr = \frac{c}{a} = 4$

$pqr = \frac{-d}{a} = 5$

$\Rightarrow \frac{8}{5}$

ii) $(pq-4p-4q+16)(r-4)$

$pqr-4pr-4qr+16r-4pq+16p+16q-64$

$pqr-4(pr+qr+pq)+16(r+p+q)-64$

$r+p+q = \frac{-b}{a} = 2$

$2-4(4)+16(5)-64 = 102$

iii) ~~$p+3$~~ $(p+q+r)^3 \Rightarrow$

$(p+q+r)(p+q+r)^2 = (p^2+pq+pr+pq+q^2+qr+rp+rq+r^2)(p+q+r)$

$(p^2+q^2+r^2+2pq+2pr+2qr)(p+q+r)$

$= \cancel{p^3} + \cancel{q^2p} + \cancel{r^2p} + \cancel{2p^2q} + \cancel{2p^2r} + 2prq$

$+ \cancel{p^2q} + \cancel{q^3} + \cancel{r^2q} + \cancel{2pq^2} + \cancel{2prq} + 2q^2r + \cancel{p^2r} + \cancel{q^2r} + \cancel{r^3} + \cancel{2pqr} + \cancel{2pr^2} + \cancel{2qr^2}$

$p^3+q^3+r^3 = (p+q+r)^3 - 6(prq) - 2(qp+pr+qr) - 3(q+p+r)$

$3q^2p + 3p^2q + 3p^2r + 3r^2q + 3r^2p + 3q^2r$

$3q(qp+qr) + 3p(pq+pr) + 3r(rq+rp)$

$(3q+3p+3r)(2(qp+qr+pr)$

$p^3+q^3+r^3 = 8-30-24-6 = -52$

5/8

Examiner Comments: (i) B1M1A1 (ii) M1A1A0 (iii) M0A0

This candidate writes down the correct values for the sum, pair sum and product, and then uses a correct identity for the sum of the reciprocals and evaluates this correctly.

In part (ii), this candidate expands the brackets and reaches a correct expression but makes an error when evaluating it.

In part (iii), the candidate does not use a correct identity and so scores no marks.

Student Response C

i) $p+q+r = -\frac{b}{a} = 2 = \Sigma p$

$\Sigma pq = \frac{c}{a} = 4$

$\Sigma pqr = -\frac{d}{a} = 5$

$$\frac{2q+2p}{pq} + \frac{2}{r}$$

$$= \frac{(2q+2p)r + 2pq}{pqr}$$

$$= \frac{2qr + 2pr + 2pq}{pqr}$$

$$= \frac{2(\Sigma pq)}{\Sigma pqr} = \frac{2 \times 4}{5} = \frac{8}{5}$$

ii) $(p-4)(q-4)(r-4) = (pq - 4q - 4p + 16)(r-4)$

$= (pq - 4q - 4p + 16)(r-4)$

$= pqr - 4pq - 4qr - 4pr + 16r + 16q$

$+ 16p - 64$

$= pqr - 4pq - 4qr - 4pr + 16r + 16q + 16p$

$- 64$

$= pqr - 4(\Sigma pq) + 16(\Sigma r) - 64$

$= 5 - 4 \times 4 + 16 \times 2 - 64$

$= 5 - 16 + 32 - 64 = -43$

iii) $p^3 + q^3 + r^3 = (p+q+r)^3 - 3\Sigma p \Sigma pq + 3\Sigma pqr$

$=$ ~~$2^3 - 3\times$~~

$2^3 - 3 \times 4 \times 2 + 3 \times 5$

$= 8 - 9 = -1$

Check

$(p+q+r)^3 = (p+q+r)(p+q+r)(p+q+r)$

$= (p^2 + pq + pr + qp + q^2 + qr + rp + r^2)(p+q+r)$

$= (p^2 + pq + pr + qp + q^2 + qr + rp + r^2)(p+q+r)$

~~$= p^3 + p^2q + p^2r + p^2q + q^2p + pqr + q^2p + q^3 + q^2r$~~

8/8

Examiner Comments: (i) B1M1A1 (ii) M1A1A1 (iii) M1A1

This candidate writes down the correct values for the sum, pair sum and product, establishes correct identities for all 3 parts and evaluates them all correctly.

Exemplar Question 3

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3

$$f(x) = \frac{1}{\sqrt{4x^2+9}}$$

(*a*) Using a substitution, that should be stated clearly, show that

$$\int f(x)\mathrm{d}x = A \sinh^{-1}(Bx) + c$$

where c is an arbitrary constant and A and B are constants to be found.

(4)

(*b*) Hence find, in exact form in terms of natural logarithms, the mean value of f(x) over the interval [0, 3].

(2)

(Total for Question 3 is 6 marks)

Mean Score 3.6 out of 6

Examiner Comments

Part (a) required an integration by substitution. Unfortunately, some students merely used the formula book without performing any substitution. Those who chose an appropriate substitution, usually $x = \frac{1}{2}u$ or $x = \frac{3}{2}\sinh u$, tended to proceed correctly. The method mark was still available to those who chose a substitution that did not lead to an easily integrable form. There were very few cases where dx was replaced with $\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}u$ rather than $\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$.

In part (b) the concept of mean value was widely known. A few errors were seen in the use of the logarithmic form of $\sinh^{-1}(x)$ but generally the two marks here were widely scored.

Mark Scheme

Question	Scheme	Marks	AOs
3(a) Way 1	$x=\frac{3}{2}\sinh u$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}}=\int \frac{1}{\sqrt{4\left(\frac{9}{4}\right)\sinh^2 u+9}}\times\frac{3}{2}\cosh u\ du$	M1	3.1a
	$=\int \frac{1}{2}\ du$	A1	1.1b
	$=\int \frac{1}{2}\ du=\frac{1}{2}u=\frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right)+c$	A1	1.1b
		(4)	
(a) Way 2	$x=\frac{3}{2}\tan u$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}}=\int \frac{1}{\sqrt{4\left(\frac{9}{4}\right)\tan^2 u+9}}\times\frac{3}{2}\sec^2 u\ du$	M1	3.1a
	$=\int \frac{1}{2}\sec u\ du$	A1	1.1b
	$=\frac{1}{2}\ln(\sec u+\tan u)=\frac{1}{2}\ln\left(\frac{2x}{3}+\sqrt{1+\left(\frac{2x}{3}\right)^2}\right)$ $u=\frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right)+c$	A1	1.1b
(a) Way 3	$x=\frac{1}{2}u$ or $x=ku$ where $k>0$ $k\neq 1$	B1	2.1
	$\int \frac{dx}{\sqrt{4x^2+9}}=\int \frac{1}{\sqrt{4\left(\frac{1}{4}\right)u^2+9}}\times\frac{1}{2}du$	M1	3.1a
	$=\frac{1}{2}\int \frac{1}{\sqrt{u^2+9}}du$ $\left(\text{or } \frac{1}{2}\int \frac{1}{\sqrt{u^2+\frac{9}{4k^2}}}du \text{ for } x=ku\right)$	A1	1.1b
	$=\frac{1}{2}\sinh^{-1}\frac{u}{3}=\frac{1}{2}\sinh^{-1}\frac{2x}{3}+c$	A1	1.1b
(b)	Mean value = $\frac{1}{3(-0)}\left[\frac{1}{2}\sinh^{-1}\left(\frac{2x}{3}\right)\right]_0^3=\frac{1}{3}\times\frac{1}{2}\sinh^{-1}\left(\frac{2\times 3}{3}\right)(-0)$	M1	2.1

	$=\frac{1}{6}\ln\left(2+\sqrt{5}\right)$ **(Brackets are required)**	A1ft	1.1b
		(2)	
			(6 marks)

Notes

(a)
B1: Selects an appropriate substitution leading to an integrable form
M1: Demonstrates a fully correct method for the substitution that includes substituting into the function and dealing with the "d*x*". The substitution being substituted does not need to be "correct" for this mark but the substitution must be an attempt at $\int \frac{1}{\sqrt{4\left[\mathrm{f}(u)\right]^2+9}}\times \mathrm{f}'(u)\,\mathrm{d}u$ with the $\mathrm{f}'(u)$ correct for their substitution. E.g. if $x=\frac{1}{2}u$ is used, must see $\mathrm{d}x=\frac{1}{2}\mathrm{d}u$ not $2\mathrm{d}u$.
A1: Correct simplified integral in terms of *u* from correct work and from a correct substitution
A1: Correct answer including "+ *c*". Allow arcsinh or arsinh for $\sinh^{-1}$ from correct work and from a correct substitution
(b)
M1: Correctly applies the method for the mean value for their integration which must be of the form specified in part (a) and substitutes the limits 0 and 3 but condone omission of 0
A1: Correct exact answer (follow through their *A* and *B*). **Brackets are required if appropriate.**

Student Response A

a) $\frac{1}{\sqrt{4x^2+9}} = \frac{1}{2\sqrt{x^2+9/4}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x^2+9/4}}$

Let $u =$

...

$\int f(x)\, dx = \frac{1}{2}\sinh^{-1}\left(\frac{2}{3}x\right) + c$

b) $\int f(x)\, dx = \frac{1}{2}\sinh^{-1}\left(\frac{2}{3}x\right) + c$

$\bar{f}(x) = \frac{1}{3} \cdot \frac{1}{2}\left[\sinh^{-1}\left(\frac{2}{3}x\right)\right]_0^3$

$= \frac{1}{6}\left(\sinh^{-1}(2) - \sinh^{-1}(0)\right)$

$= \frac{1}{6} \cdot \ln(2+\sqrt{5})$

2/6

Examiner Comments: (a) B0M0A0A0 (b) M1A1

In part (a), this candidate does not use a substitution and so no marks are available.

In part (b), the candidate uses their correct answer from (b) and obtains the correct mean value.

Student Response B

a. let $x = \frac{3}{2}\sinh\theta$

$dx = \frac{3}{2}\cosh\theta \, d\theta$

$$\int f(x) = \int \frac{1}{\sqrt{4x^2+9}}\,dx$$

$$= \int \frac{1}{\sqrt{4(\frac{3}{2}\sinh\theta)^2+9}} \cdot \frac{3}{2}\cosh\theta \, d\theta$$

$$= \int \frac{1}{\sqrt{9\cosh^2\theta}} \cdot \frac{3}{2}\cosh\theta \, d\theta$$

$$= \int \frac{1}{3\cosh\theta} \cdot \frac{3}{2}\cosh\theta \, d\theta$$

$$= \int \frac{1}{2}\,d\theta$$

$\frac{2x}{3} = \sinh\theta$

$\operatorname{arsinh}\frac{2x}{3} = \theta$

$$= \frac{1}{2}\theta + c = \frac{1}{2}\sinh^{-1}\left(\frac{2}{3}x\right) + c$$

b. $\int_0^3 F(x)\,dx$

$= \left[\frac{1}{2}\sinh^{-1}\left(\frac{2}{3}x\right)\right]_0^3$

$= \frac{1}{2}\sinh^{-1}(2) - \frac{1}{2}\sinh^{-1}(0)$

$= \frac{1}{2}\ln\left(2 + \sqrt{2^2+1}\right)$

$= \frac{1}{2}\ln\left(2+\sqrt{5}\right)$

4/6

Examiner Comments: (a) B1M1A1A1 (b) M1A1

In part (a), this candidate uses an appropriate substitution and proceeds to obtain the correct answer.

In part (b), the candidate omits the "$\frac{1}{3-0}$" for the mean value and so score no marks.

Student Response C

a) let $x = \frac{3}{2}\tan u$.

$\frac{dx}{du} = \frac{3}{2}\sec^2 u$.

$dx = \frac{3}{2}\sec^2 u\,du$

$\int f(x)\,dx = \int \frac{\frac{3}{2}\sec^2 u}{\sqrt{4(\frac{3}{2}\tan u)^2+9}}\,du$.

$= \int \frac{\frac{3}{2}\sec^2 u\,du}{\sqrt{9\tan^2 u+9}} = \int \frac{\frac{3}{2}\sec^2 u}{3\sec u}\,du$.

Aside → $[1 + \tan^2 u = \sec^2 u.]$

$\int \frac{1}{2\cos u}\,du = \frac{1}{2}\ln|\sec u + \tan u| + c$.

Aside → $x = \frac{3}{2}\tan u \therefore \tan u = \frac{2}{3}x$.

$\sec u = \sqrt{1 + (\frac{2}{3}x)^2}$

$$\therefore I = \frac{1}{2}\ln\left|\frac{2}{3}x + \sqrt{1+\left(\frac{2}{3}x\right)^2}\right| + c$$

$$= \frac{1}{2}\sinh^{-1}\left(\frac{2}{3}x\right) + c.$$

$$A = \frac{1}{2}, \qquad B = \frac{2}{3}.$$

b.) $$\int_0^3 f(x) = \left[\frac{1}{2}\sinh^{-1}\left(\frac{2}{3}x\right)\right]_0^3.$$

$$= \frac{1}{2}\sinh^{-1}(2) - \frac{1}{2}\sinh^{-1}(0).$$

$$= \frac{1}{2}\sinh^{-1}(2)$$

$$= \frac{1}{2}\ln\left|2+\sqrt{1+(2)^2}\right|$$

$$= \frac{1}{2}\ln\left|2+\sqrt{5}\right|$$

$$\bar{f} = \frac{1}{3-0}\left(\frac{1}{2}\ln|2+\sqrt{5}|\right).$$

$$= \frac{1}{6}\ln(2+\sqrt{5}|.$$

6/6

Examiner Comments: (a) B1M1A1A1 (b) M1A1

In part (a), this candidate uses an appropriate substitution and proceeds to obtain the correct answer.

In part (b), the candidate uses a correct strategy and scores both marks.

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Exemplar Question 4

4 The infinite series C and S are defined by

$$C = \cos\theta + \frac{1}{2}\cos 5\theta + \frac{1}{4}\cos 9\theta + \frac{1}{8}\cos 13\theta + \ldots$$

$$S = \sin\theta + \frac{1}{2}\sin 5\theta + \frac{1}{4}\sin 9\theta + \frac{1}{8}\sin 13\theta + \ldots$$

Given that the series C and S are both convergent,

(*a*) show that

$$C + iS = \frac{2e^{i\theta}}{2-e^{4i\theta}}$$

(4)

(*b*) Hence show that

$$S = \frac{4\sin q + 2\sin 3q}{5 - 4\cos 4q}$$

(4)

(Total for Question 4 is 8 marks)

Mean Score 4.2 out of 8

Examiner Comments

In part (a), obtaining C + iS as an exponential series was widely achieved. The majority were also able to use the sum to infinity formula to obtain the given answer. A few attempted to use the sum to *n* terms formula for a geometric series.

Part (b) proved tough for all but the most confident students. Incorrect attempts included multiplying numerator and denominator by $e^{\pm 4i\theta}$, $2 + e^{-4i\theta}$ or $2 - e^{4i\theta}$ rather than the required $2 - e^{-4i\theta}$. Those who knew the correct strategy usually obtained a correct expression and invariably went on to revert to trigonometric form and reach the given answer. The alternative of converting to trigonometric form and then rationalising was successful for some but there were often slips in the multiplications and some were unable to use the correct addition formula to reach the printed answer.

Mark Scheme

Question	Scheme	Marks	AOs
4(a) Way 1	$\mathrm{C}+\mathrm{iS}=\cos\theta+\mathrm{i}\sin\theta+\frac{1}{2}(\cos5\theta+\mathrm{i}\sin5\theta)\left(+\frac{1}{4}(\cos9\theta+\mathrm{i}\sin9\theta)+\ldots\right)$	M1	1.1b
	$=\mathrm{e}^{\mathrm{i}\theta}+\frac{1}{2}\mathrm{e}^{5\mathrm{i}\theta}\left(+\frac{1}{4}\mathrm{e}^{9\mathrm{i}\theta}+\ldots\right)$	A1	2.1
	$\mathrm{C}+\mathrm{iS}=\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{1-\frac{1}{2}\mathrm{e}^{4\mathrm{i}\theta}}$	M1	3.1a
	$=\dfrac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}$ *	A1*	1.1b
		(4)	
(a) Way 2	$\mathrm{C}+\mathrm{iS}=\cos\theta+\mathrm{i}\sin\theta+\frac{1}{2}(\cos5\theta+\mathrm{i}\sin5\theta)\left(+\frac{1}{4}(\cos9\theta+\mathrm{i}\sin9\theta)+\ldots\right)$	M1	1.1b
	$\mathrm{C}+\mathrm{iS}=\cos\theta+\mathrm{i}\sin\theta+\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)^5\left(+\frac{1}{4}(\cos\theta+\mathrm{i}\sin\theta)^9+\ldots\right)$	A1	2.1
	$\mathrm{C}+\mathrm{iS}=\dfrac{\cos\theta+\mathrm{i}\sin\theta}{1-\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)^4}=\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{1-\frac{1}{2}\mathrm{e}^{4\mathrm{i}\theta}}$	M1	3.1a
	$=\dfrac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}$ *	A1*	1.1b
		(4)	
(b) Way 1	$\dfrac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}\times\dfrac{2-\mathrm{e}^{-4\mathrm{i}\theta}}{2-\mathrm{e}^{-4\mathrm{i}\theta}}$	M1	3.1a
	$\dfrac{4\mathrm{e}^{\mathrm{i}\theta}-2\mathrm{e}^{-3\mathrm{i}\theta}}{4-2\mathrm{e}^{-4\mathrm{i}\theta}-2\mathrm{e}^{4\mathrm{i}\theta}+1}$	A1	1.1b
	$\dfrac{4\cos\theta+4\mathrm{i}\sin\theta-2\cos3\theta+2\mathrm{i}\sin3\theta}{5-2\cos4\theta+2\mathrm{i}\sin4\theta-2\cos4\theta-2\mathrm{i}\sin4\theta}$ **Dependent on the first M**	**d**M1	2.1
	$S=\dfrac{4\sin\theta+2\sin3\theta}{5-4\cos4\theta}$ *	A1*	1.1b
		(4)	
(b) Way 2	$\dfrac{2\mathrm{e}^{\mathrm{i}\theta}}{2-\mathrm{e}^{4\mathrm{i}\theta}}=\dfrac{2(\cos\theta+\mathrm{i}\sin\theta)}{2-(\cos4\theta+\mathrm{i}\sin4\theta)}\times\dfrac{2-(\cos4\theta-\mathrm{i}\sin4\theta)}{2-(\cos4\theta-\mathrm{i}\sin4\theta)}$	M1	3.1a
	$\dfrac{4\cos\theta+4\mathrm{i}\sin\theta-2\cos\theta\cos4\theta-2\sin\theta\sin4\theta+2\mathrm{i}\sin4\theta\cos\theta-2\mathrm{i}\sin\theta\cos4\theta}{4+\cos^2 4\theta+\sin^2 4\theta-4\cos4\theta}$	A1	1.1b
	$\dfrac{4\cos\theta+4\mathrm{i}\sin\theta-2\cos3\theta+2\mathrm{i}\sin3\theta}{5-2\cos4\theta+2\mathrm{i}\sin4\theta-2\cos4\theta-2\mathrm{i}\sin4\theta}$ **Dependent on the first M**	**d**M1	2.1
	$S=\dfrac{4\sin\theta+2\sin3\theta}{5-4\cos4\theta}$ *	A1*	1.1b
		(8 marks)	

Notes
(a) **Way 1** M1: Combines the two series by pairing the multiples of θ (At least up to 5θ) A1: Converts to Euler form correctly (At least up to 5θ) M1: Recognises that C + iS is a convergent geometric series and uses the sum to infinity of a GP A1*: Reaches the printed answer with no errors **Way 2** M1: Combines the two series by pairing the multiples of θ (At least up to 5θ) A1: Converts to power form correctly (At least up to 5θ) M1: Recognises that C + iS is a convergent geometric series and uses the sum to infinity of a GP A1*: Reaches the printed answer with no errors (b) **Way 1** M1: Multiplies numerator and denominator by $2-e^{-4i\theta}$ A1: Correct fraction in terms of exponentials **d**M1: Converts back to trigonometric form A1*: Reaches the printed answer with no errors **Way 2** M1: Converts back to trigonometric form and realises the need to make the denominator real and multiplies numerator and denominator by the complex conjugate of the denominator which is **correct** for their fraction A1: Correct fraction in terms of trigonometric functions **d**M1: Uses the correct addition formula to obtain sin 3θ in the numerator A1*: Reaches the printed answer with no errors

Student Response A

$C + iS = (\cos\theta + i\sin\theta) + \frac{1}{2}(\cos 5\theta + i\sin 5\theta) + \frac{1}{4}(\cos 9\theta + i\sin 9\theta) + \frac{1}{8}(\cos 13\theta + i\sin 13\theta)$

$= e^{i\theta} + \frac{1}{2}e^{5i\theta} + \frac{1}{4}e^{9i\theta} + \frac{1}{8}e^{13i\theta}$

$= e^{i\theta}\left(1 + \frac{1}{2}e^{4i\theta} + \frac{1}{4}e^{8i\theta} + \frac{1}{8}e^{12i\theta}\right)$

$\sum_{0}^{n-1} wz^{r-1} = \frac{w(z^n - 1)}{z^n - 1}$ $\quad w = e^{i\theta}$, $r = \frac{1}{2}e^{4i\theta}$

$= e^{i\theta}($

b) $\operatorname{Im}(C + iS) = S$

$\frac{2e^{i\theta}}{2 - e^{4i\theta}} = \frac{2(\cos\theta + i\sin\theta)}{2 - (\cos 4\theta + i\sin 4\theta)}$

$= \frac{2i\sin\theta}{2 + i\sin 4\theta}$

2/8

Examiner Comments: (a) M1A1M0A0 (b) M0A0M0A0

This candidate scores the first 2 marks in part (a) for pairing the terms correctly and changing to exponential form. However no further progress is made as the candidate does not apply the sum to infinity formula for a GP.

In part (b), this candidate converts the given expression from part (a) to trigonometric form but does not multiply the numerator and denominator by the complex conjugate of the denominator and so scores no marks in this part.

Student Response B

a) $C + iS = (\cos\theta + i\sin\theta) + \frac{1}{2}(\cos 5\theta + i\sin 5\theta) + \frac{1}{4}(\cos 9\theta + i\sin 9\theta) + \ldots$

$= e^{i\theta} + \frac{1}{2}e^{5i\theta} + \frac{1}{4}e^{9i\theta}$

$r =$ ~~difference~~ ratio between $e^{i\theta}$ and $\frac{1}{2}e^{5i\theta}$ $= \frac{1}{2}e^{4i\theta}$

$S_\infty = \frac{e^{i\theta}}{1 - \frac{1}{2}e^{4i\theta}}$

$S_\infty = \frac{a}{1-r}$

$= \frac{2e^{i\theta}}{2 - e^{4i\theta}}$

b) $C + iS = \frac{2\cos\theta + i\sin\theta}{2 - (\cos 4\theta + i\sin 4\theta)}$

$\frac{(2e^{i\theta}) \times (2 + e^{4i\theta})}{(2 - e^{4i\theta}) \times (2 + e^{4i\theta})} = \frac{4e^{i\theta} + 2e^{5i\theta}}{4 - e^{8i\theta}}$

~~taking sins only $= 4\sin\theta + 2$~~

$= \frac{4\cos\theta + i\sin\theta + 2\cos 5\theta + i\sin 5\theta}{4 - (\cos 8\theta + i\sin 8\theta)}$

4/8

Examiner Comments: (a) M1A1M1A1 (b) M0A0M0A0

In part (a), this candidate pairs the terms correctly and changes to exponential form. The sum to infinity formula is then applied with the correct first term and common ratio to achieve the printed answer.

In part (b), this candidate does not make any valid progress as they do not multiply the numerator and denominator by the complex conjugate of the denominator.

Student Response C

a)
$$C + iS = \cos\theta + i\sin\theta + \tfrac{1}{2}(\cos 5\theta + i\sin 5\theta) + \tfrac{1}{4}(\cos 9\theta + i\sin 9\theta) + \ldots$$
$$= e^{i\theta} + \tfrac{1}{2}e^{5i\theta} + \tfrac{1}{4}e^{9i\theta} + \ldots$$
$$= e^{i\theta} + e^{i\theta}\left(\tfrac{1}{2}e^{4i\theta}\right) + e^{i\theta}\left(\tfrac{1}{4}e^{8i\theta}\right) + \ldots$$
$$= e^{i\theta} + e^{i\theta}\left(\tfrac{1}{2}e^{4i\theta}\right)^1 + e^{i\theta}\left(\tfrac{1}{2}e^{4i\theta}\right)^2 + \ldots$$
$$= \sum_{r=0}^{\infty} e^{i\theta}\left(\tfrac{1}{2}e^{4i\theta}\right)^r$$
$$= \frac{e^{i\theta}}{1 - \frac{1}{2}e^{4i\theta}} \times \frac{2}{2}$$
$$= \frac{2e^{i\theta}}{2 - e^{4i\theta}} \quad \square$$

b)
$$\therefore C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}} \times \frac{2 - e^{-4i\theta}}{2 - e^{-4i\theta}}$$
$$= \frac{4e^{i\theta} - 2e^{-3i\theta}}{4 - 2e^{4i\theta} - 2e^{-4i\theta} + 1}$$
$$= \frac{4e^{i\theta} - 2e^{-3i\theta}}{5 - 2(e^{4i\theta} + e^{-4i\theta})}$$
$$= \frac{4\cos\theta + 4i\sin\theta - 2\cos(-3\theta) - 2i\sin(-3\theta)}{5 - 2(2\cos 4\theta)}$$
$$= \frac{4\cos\theta + 4i\sin\theta - 2\cos 3\theta + 2i\sin 3\theta}{5 - 4\cos 4\theta}$$

By comparing imaginary terms:
$$\therefore S = \frac{4\sin\theta + 2\sin 3\theta}{5 - 4\cos 4\theta} \quad \square$$

8/8

Examiner Comments: (a) M1A1M0A1 (b) M1A1M1A1

In part (a), this candidate pairs the terms correctly and changes to exponential form. The sum to infinity formula is then applied with the correct first term and common ratio to achieve the printed answer. (Note that the work that has not been crossed out has been marked)

In part (b), the candidate multiplies the numerator and denominator by the correct complex conjugate of the denominator and reaches the printed answer with no errors.

Exemplar Question 5

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5 An engineer is investigating the motion of a sprung diving board at a swimming pool.

Let E be the position of the end of the diving board when it is at rest in its equilibrium position and when there is no diver standing on the diving board.

A diver jumps from the diving board.
The vertical displacement, h cm, of the end of the diving board above E is modelled by the differential equation

$$4\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + 4\frac{\mathrm{d}h}{\mathrm{d}t} + 37h = 0$$

where t seconds is the time after the diver jumps.

(*a*) Find a general solution of the differential equation.

(2)

When $t = 0$, the end of the diving board is 20 cm below E and is moving upwards with a speed of 55 cm s^{-1}.

(*b*) Find, according to the model, the maximum vertical displacement of the end of the diving board above E.

(8)

(*c*) Comment on the suitability of the model for large values of t.

(2)

(Total for Question 5 is 12 marks)

Mean Score 6.8 out of 12

Examiner Comments

Question 5 featured a model involving a second order differential equation and the latter marks in part (b) were not widely scored.

In part (a), most formed and solved the auxiliary equation correctly although occasionally m was obtained as $-\frac{1}{2} \pm 6\text{i}$ rather than $-\frac{1}{2} \pm 3\text{i}$. The correct form of the general solution was widely seen although the "$h =$" was sometimes missing or y or x were used instead of h and t.

In part (b), most students were able to appropriately obtain a value for both constants. Common errors were to set $h = 20$ rather than -20 and to not use the product rule when differentiating h. Finding the maximum proved challenging and many had an incorrect strategy. A significant number of students attempted to apply $R\sin(3t + \alpha)$ to the trigonometric part of their h instead of their derivative, leading to answers of R or $R\text{e}^{-0.5t}$. The more sensible route of using $\frac{\sin 3t}{\cos 3t}$ to get an equation in tan $3t$ saw more success. Those who obtained the correct equation often failed to obtain the smallest positive value of t. Some just dropped the minus sign from the calculator value of $\tan^{-1}\left(-\frac{22}{21}\right)$. A small number did not go on to obtain h for their t. Occasionally, work in degrees was seen, often producing clearly unreasonable values for h_{max}.

Part (c) required students to comment on the suitability of the model for large values of t and this was well answered on the whole. Most deduced that h tended to zero as $t \to \infty$ and were able to make a sensible comment which was often perceptive about the mechanics of the situation. A few however, did not offer any appraisal of the model's suitability in their answer.

Mark Scheme

Question	Scheme	Marks	AOs
5(a)	$4m^2+4m+37=0 \Rightarrow m=-\frac{1}{2}\pm 3\text{i}$	M1	1.1b
	$h=\text{e}^{-0.5t}\left(A\cos 3t+B\sin 3t\right)$	A1	1.1b
		(2)	
(b)	$t=0,\ h=-20 \Rightarrow A=-20$	M1	3.4
	$\frac{\text{d}h}{\text{d}t}=-0.5\text{e}^{-0.5t}\left(A\cos 3t+B\sin 3t\right)+\text{e}^{-0.5t}\left(-3A\sin 3t+3B\cos 3t\right)$ $t=0,\ \frac{\text{d}h}{\text{d}t}=55 \Rightarrow B=...$ (NB B = 15)	M1	3.4
	$(h=)\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)$	A1	1.1b
	$-0.5\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)+\text{e}^{-0.5t}\left(60\sin 3t+45\cos 3t\right)=0$ or e.g. $-0.5\text{e}^{-0.5t}\left(15\sin 3t-20\cos 3t\right)+\frac{25\sqrt{37}}{2}\text{e}^{-0.5t}\sin\left(3t+\arctan\frac{22}{21}\right)=0$ $\Rightarrow t=...$	M1	3.1b
	$\tan 3t=-\frac{22}{21}$ or e.g. $3t+\tan^{-1}\frac{22}{21}=0$	A1 **M1 on ePEN**	2.1
	t = 0.778 s	A1	1.1b
	$h=\text{e}^{-0.5\times\text{"}0.778\text{"}}\left(15\sin\left(3\times\text{"}0.778\text{"}\right)-20\cos\left(3\times\text{"}0.778\text{"}\right)\right)$	**d**M1	1.1b
	= 16.7 cm	A1	3.2a
		(8)	
(c)	E.g. considers large values of t in the model for h **or** states that for large values of t, h becomes smaller or becomes zero	M1	3.4
	E.g. • The value of h is very small when t is large and this is likely to be correct (as the displacement of end of the board should get smaller and smaller) • This suggests the model is suitable • This is realistic • This is suitable as the board will tend towards its equilibrium position • When t is large the value of h is never zero so the model is not really appropriate for large values of t	A1 **B1 on ePEN**	3.2b
		(2)	
			(12 marks)

Notes

(a)

M1: Uses the model to form and solve the auxiliary equation $4m^2+4m+37=0$

See General Guidance for awarding this mark. This can be implied by correct values for m (from calculator)

A1: Correct general solution including "h ="

(b)
M1: Uses the model and the initial conditions to establish the value of "*A*". Need to see $t = 0$ and $h = \pm 20$ leading to a value for "*A*". This may be implied by $A = -20$ or $A = 20$.
M1: Differentiates their model using the product rule and uses the initial conditions, $t = 0$ with $\frac{\mathrm{d}h}{\mathrm{d}t} = \pm 55$, to establish the value of "*B*"
A1: Correct particular solution or correct values for *A* and *B*
M1: Uses their solution to the model with a correct strategy to obtain a value for *t* e.g. differentiates or uses their derivative from earlier, sets equal to zero and solves for *t*
A1Correct equation for *t*
A1: Correct value for *t* (allow awrt 0.778 if necessary) but this value may be implied.
dM1: Uses the model and their **positive** value for *t* to find the maximum displacement - **if their *t* is incorrect there must be some indication that they are using their *h* and not just a number written down. E.g. must see substitution into their *h* or they re-state their *h* and obtain a value for *h*.**
Dependent on all the previous method marks
A1: Correct value (awrt 16.7 (units not needed))
(c)
M1: Considers the model for large values of *t* either by substituting values or by considering the expression and commenting on its behaviour for large values of *t*. E.g. as $t \to \infty,\ h \to 0$ or as $t \to \infty,\ e^{-0.5t} \to 0$ or as $t \to \infty$ the oscillations become smaller etc.
A1: Makes a suitable comment – see scheme for examples

Student Response A

$4m^2+4m+37=0$

$m=\frac{-1+6i}{2}$ $m=\frac{-1-6i}{2}$

$E = A - \frac{1}{2}(A\cos\frac{6}{2}t\ B i\sin(\frac{6}{2}t)$

$E \quad -\frac{1}{2}(A\cos(\frac{6}{2}t) - Bi\sin(\frac{6}{2}t))$

$t=0, \quad h=20, \quad E=55$

$20 = -\frac{1}{2}(A\cos)$

$A = -40$

$55 = -\frac{1}{2}(-40\cdot\frac{6}{2}\cdot -\sin\frac{6}{2}t - Bi\frac{6}{2}\cos\frac{6}{2}t)$

$-110 = -\frac{6}{2}B$

$220 = 6B$

$B = 110/3$

Max occurs when B and A are at their greatest -ive value

Max $E = -\frac{1}{2}(-40 - 110/3) = 115/3$

large values of t would not be suitable as sin and cos repeat themselves, are periodic eg. 180 seconds would = 0, for sin but 181 would be greater than zero.

2/12

Examiner Comments: (a) M1A0 (b) M1M0A0M0A0A0M0A0 (c) M0A0

It is worth noting the lack of labelling in this student’s response. In this case, the work is marked in the order it is presented.

For part (a), this candidate solves the correct auxiliary equation but does not use the roots to form the correct general solution.

For part (b), the candidate uses the 20 with $t = 0$ in an attempt to find their constant A and this scores the first method mark. As there is no subsequent attempt to apply the product rule to their general solution and no appropriate strategy to find a value for t for the maximum displacement, no more marks are available.

For part (c), there is no relevant comment relating to the suitability of the model for large values of t.

Student Response B

5a) auxiliary equation:

$4m^2 + 4m + 37 = 0 \rightarrow m = -0.5 \pm 3i$

$\therefore h = e^{-0.5t}(A\cos 3t + B\sin 3t)$

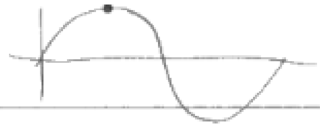
b) $t = 0,\ h = -20,\ \frac{dh}{dt} = 55$

$-20 = A$

$\frac{dh}{dt} = e^{-0.5t}(-3A\sin 3t + 3B\cos 3t) - e^{-0.5t}(A\cos 3t + B\sin 3t)$

$= e^{-0.5t}((-3A - B)\sin 3t + (3B - A)\cos 3t)$

$\frac{dh}{dt} = e^{-0.5t}((60 - B)\sin 3t + (3B + 20)\cos 3t)$



$55 = 3B + 20 \rightarrow B = \frac{35}{3}$

so $h = e^{-0.5t}\left(\frac{35}{3}\sin 3t - 20\cos 3t\right)$

period $= \frac{2\pi}{3} \rightarrow$ max when $t = \frac{T}{4} = \frac{2\pi}{12}$

$\frac{35}{3}\sin 3t - 20\cos 3t = R\sin(3t + \alpha)$

$t = \frac{2\pi}{12} \Rightarrow h = e^{-0.5(2\pi/12)} \times 7 = 5.388\text{ m}$

c) For large values of t, $e^{-0.5t} \rightarrow 0$, which is accurate; the max. amplitude of oscillations decreases to zero exponentially.

So the model is suitable.

6/12

Examiner Comments: (a) M1A1 (b) M1M1A0M0A0A0M0A0 (c) M1A1

In part (a), this candidate solves the correct auxiliary equation and forms the correct general solution.

In part (b), the candidate uses the 20 with $t = 0$ in an attempt to find their constant A (implied by $A = -20$) and this scores the first method mark. The candidate then applies the product rule to their general solution and uses the 55 in an attempt to find their constant B and this scores the second method mark. The subsequent strategy to find the maximum displacement is not correct and no more marks are available.

For part (c), there is a relevant comment relating to the suitability of the model for large values of t and a conclusion regarding the suitability of the model and so both marks are scored.

Student Response C

a. AE: $4m^2 + 4m + 37 = 0$

$m = -\frac{1}{2} \pm 3i$

$\therefore h = e^{-\frac{1}{2}t}(A\cos 3t + B\sin 3t)$

b. $\frac{dh}{dt} = -\frac{1}{2}e^{-\frac{1}{2}t}(A\cos 3t + B\sin 3t)$

$+ e^{-\frac{1}{2}t}(-3A\sin 3t + 3B\cos 3t)$

$t=0,\ h=-20,\ \frac{dh}{dt} = 55$

$h:\ -20 = e^{0}(A\cos 0 + B\sin 0)$

$-20 = A$

$\frac{dh}{dt}:\ 55 = -\frac{1}{2}(A) + (3B)$

$55 = 10 + 3B$

$45 = 3B$

$15 = B$

$h = e^{-\frac{1}{2}t}(-20\cos 3t + 15\sin 3t)$

$\frac{dh}{dt} = -\frac{1}{2}e^{-\frac{1}{2}t}(-20\cos 3t + 15\sin 3t)$

$+ e^{-\frac{1}{2}t}(60\sin 3t + 45\cos 3t)$

$= e^{-\frac{1}{2}t}(10\cos 3t - \frac{15}{2}\sin 3t + 60\sin 3t + 45\cos 3t)$

$= e^{-\frac{1}{2}t}\left(55\cos 3t + \frac{105}{2}\sin 3t\right)$

$\frac{dh}{dt} = 0$: $55\cos 3t = -\frac{105}{2}\sin 3t$

$110\cos 3t = -105\sin 3t$

$\frac{-110}{105} = \tan 3t$

$0.778 = t$

$h = e^{-\frac{1}{2}(0.778)}\left(-20\cos(3\times 0.778) + 15\sin(0.778\times 3)\right)$

$h = 16.7$ cm.

C. Seems quite suitable as $h \to 0$ as $t \to \infty$ as the $e^{-\frac{1}{2}t}$ term tends to 0. This makes sense as the board is returning to the equilibrium position.

12/12

Examiner Comments: (a) M1A1 (b) M1M1A1M1A1A1M1A1 (c) M1A1

In part (a), this candidate solves the correct auxiliary equation and forms the correct general solution.

In part (b), the candidate uses the 20 with $t = 0$ in an attempt to find their constant A and then applies the product rule to their general solution and uses the 55 in an attempt to find their constant *B* correctly. The subsequent strategy to find the maximum displacement is fully correct and so full marks are scored in part (b).

For part (c), there is a relevant comment relating to the suitability of the model for large values of *t* and a conclusion regarding the suitability of the model and so both marks are scored.

Exemplar Question 6

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6 In an Argand diagram, the points A, B and C are the vertices of an equilateral triangle with its centre at the origin. The point A represents the complex number $6 + 2\mathrm{i}$.

(*a*) Find the complex numbers represented by the points B and C, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact.

(6)

The points D, E and F are the midpoints of the sides of triangle ABC.

(*b*) Find the exact area of triangle DEF.

(3)

(Total for Question 6 is 9 marks)

Mean Score 2.7 out of 9

Examiner Comments

Question 6 required students to use complex roots to solve a geometric problem. It was clear that a considerable number were poorly prepared for such a task and the simplest route – to multiply 6 + 2i by the complex cube roots of unity – was not widely seen. Those who knew this method usually emerged with all six marks although a few sign slips occurred. On occasion $\omega = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$ rather than $\omega = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$ was used. Matrix methods were rare but usually correct. Of the remaining students who made a significant attempt, most knew that they had to add $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ (or $-\frac{2\pi}{3}$) to the argument of 6 + 2i but most could only deliver a decimal answer at best. Weaker attempts included the reflection of (6, 2) in the coordinate axes.

Part (b) was often not attempted by students who were unable to progress in (a) although full marks were still possible if the area of triangle AOB was found and this was a reliable route. A wide range of methods were seen, although those using the coordinates of B and C often fell foul of errors handling the surds. Some candidates got into difficulty with approaches that used $\frac{1}{2} \times$ base $\times$ perpendicular height rather than $\frac{1}{2}ab\sin C$.

Mark Scheme

Question	Scheme	Marks	AOs
6(a)	Examples: $\begin{pmatrix}\cos 120 & -\sin 120\\ \sin 120 & \cos 120\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix}=\ldots$ **or** $(6+2\text{i})\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\text{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{2\pi}{3}\right)+\text{i}\sin\left(\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)+\text{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{2\pi}{3}\right)\right)$ **or** $\sqrt{40}\text{e}^{\text{i}\arctan\left(\frac{2}{6}\right)}\text{e}^{\text{i}\left(\frac{2\pi}{3}\right)}$	M1	3.1a
	$\left(-3-\sqrt{3}\right)$ **or** $\left(3\sqrt{3}-1\right)\text{i}$	A1	1.1b
	$\left(-3-\sqrt{3}\right)+\left(3\sqrt{3}-1\right)\text{i}$	A1	1.1b
	Examples: $\begin{pmatrix}\cos 240 & -\sin 240\\ \sin 240 & \cos 240\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix}=\ldots$ **or** $(6+2\text{i})\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}\right)$ **or** $\sqrt{40}\left(\cos\arctan\left(\frac{2}{6}\right)+\text{i}\sin\arctan\left(\frac{2}{6}\right)\right)\left(\cos\left(\frac{4\pi}{3}\right)+\text{i}\sin\left(\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\left(\cos\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)+\text{i}\sin\left(\arctan\left(\frac{2}{6}\right)+\frac{4\pi}{3}\right)\right)$ **or** $\sqrt{40}\text{e}^{\text{i}\arctan\left(\frac{2}{6}\right)}\text{e}^{\text{i}\left(\frac{4\pi}{3}\right)}$	M1	3.1a
	$\left(-3+\sqrt{3}\right)$ **or** $\left(-3\sqrt{3}-1\right)\text{i}$	A1	1.1b
	$\left(-3+\sqrt{3}\right)+\left(-3\sqrt{3}-1\right)\text{i}$	A1	1.1b
		(6)	
(b) Way 1	Area ABC = $3\times\frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$ or Area AOB = $\frac{1}{2}\sqrt{6^2+2^2}\sqrt{6^2+2^2}\sin 120°$	M1	2.1
	Area DEF = $\frac{1}{4}ABC$ or $\frac{3}{4}AOB$	**d**M1	3.1a
	$=\frac{3}{8}\times 40\times\frac{\sqrt{3}}{2}=\frac{15\sqrt{3}}{2}$	A1	1.1b
		(3)	

(b) Way 2	$D\left(\frac{3-\sqrt{3}}{2},\frac{3\sqrt{3}+1}{2}\right)$ $OD=\sqrt{\left(\frac{3-\sqrt{3}}{2}\right)^2+\left(\frac{3\sqrt{3}+1}{2}\right)^2}=\sqrt{10}$ Area $DOF=\frac{1}{2}\sqrt{10}\sqrt{10}\sin 120°$	M1	2.1
	Area $DEF=3DOF$	**d**M1	3.1a
	$=3\times\frac{1}{2}\times\sqrt{10}\sqrt{10}\times\frac{\sqrt{3}}{2}=\frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 3	$AB=\sqrt{\left(9+\sqrt{3}\right)^2+\left(3-3\sqrt{3}\right)^2}=\sqrt{120}$ Area $ABC=\frac{1}{2}\sqrt{120}\sqrt{120}\sin 60°\left(=30\sqrt{3}\right)$	M1	2.1
	Area $DEF=\frac{1}{4}ABC$	**d**M1	3.1a
	$=\frac{1}{4}\times 30\sqrt{3}=\frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 4	$D\left(\frac{3-\sqrt{3}}{2},\frac{3\sqrt{3}+1}{2}\right),\ E(-3,-1),\ F\left(\frac{3+\sqrt{3}}{2},\frac{-3\sqrt{3}+1}{2}\right)$ $DE=\sqrt{\left(\frac{3-\sqrt{3}}{2}+3\right)^2+\left(\frac{3\sqrt{3}+1}{2}+1\right)^2}\ \left(=\sqrt{30}\right)$ Area $DEF=\frac{1}{2}\sqrt{30}\sqrt{30}\sin 60°$	M1 **d**M1	2.1 3.1a
	$=\frac{15\sqrt{3}}{2}$	A1	1.1b
(b) Way 5	Area $ABC=\frac{1}{2}\begin{vmatrix}6 & -3-\sqrt{3} & \sqrt{3}-3 & 6\\ 2 & 3\sqrt{3}-1 & -3\sqrt{3}-1 & 2\end{vmatrix}=30\sqrt{3}$	M1	2.1
	Area $DEF=\frac{1}{4}ABC$	**d**M1	3.1a
	$=\frac{1}{4}\times 30\sqrt{3}=\frac{15\sqrt{3}}{2}$	A1	1.1b
			(9 marks)

Notes

(a)

M1: Identifies a suitable method to rotate the given point by 120° (or equivalent) about the origin. May see equivalent work with modulus/argument or exponential form e.g. an attempt to multiply by $\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}$ or $\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$

A1: Correct real part or correct imaginary part

A1: Completely correct complex number

M1: Identifies a suitable method to rotate the given point by 240° (or equivalent e.g. rotate their *B* by 120°) about the origin

May see equivalent work with modulus/argument or exponential form e.g. an attempt to multiply $6 + 2\text{i}$ by $\cos\frac{4\pi}{3} + \text{i}\sin\frac{4\pi}{3}$ or $\text{e}^{\frac{4\pi}{3}\text{i}}$ or their *B* by $\cos\frac{2\pi}{3} + \text{i}\sin\frac{2\pi}{3}$ or $\text{e}^{\frac{2\pi}{3}\text{i}}$
A1: Correct real part or correct imaginary part
A1: Completely correct complex number
(b)
In general, the marks in (b) should be awarded as follows:
M1: Attempts to find the area of a relevant triangle
dM1: completes the problem by multiplying by an appropriate factor to find the area of *DEF*
Dependent on the first method mark
A1: Correct exact area
In some cases it may not be possible to distinguish the 2 method marks. In such cases they can be awarded together for a direct method that finds the area of *DEF*

Examples:

Way 1
M1: A correct strategy for the area of a relevant triangle such as *ABC* or *AOB*
dM1: Completes the problem by linking the area of *DEF* correctly with *ABC* or with *AOB*
A1: Correct value
Way 2
M1: A correct strategy for the area of a relevant triangle such as *DOF*
dM1: Completes the problem by linking the area of *DEF* correctly with *DOF*
A1: Correct value
Way 3
M1: A correct strategy for the area of a relevant triangle such as *ABC*
dM1: Completes the problem by linking the area of *DEF* correctly with *ABC*
A1: Correct value
Way 4
M1**d**M1: A correct strategy for the area of *DEF*. Finds 2 midpoints and attempts one side of *DEF* and uses a correct triangle area formula. By implication this scores both M marks.
A1: Correct value
Way 5
M1: A correct strategy for the area of *ABC* using the “shoelace” method.
dM1: Completes the problem by linking the area of *DEF* correctly with *ABC*
A1: Correct value
Note the marks in (b) can be scored using inexact answers from (a) and the A1 scored if an exact area is obtained.

Student Response A

a) $|6+2i| = 2\sqrt{10}$

$\arg(6+2i) = \arctan(\frac{2}{6}) = \arctan(\frac{1}{3}) = 0.322$

let $\tan\theta = \frac{1}{3}$, $\therefore$ A is represented by $2\sqrt{10}\,e^{i\theta} = z$

$w = e^{\frac{2i\pi}{3}}$

(rotating A by $\frac{2\pi}{3}$ about the origin)

B is at $zw = 2\sqrt{10}e^{i\theta}\left(e^{\frac{2i\pi}{3}}\right)$

$= 2\sqrt{10}\,e^{i(\theta+\frac{2\pi}{3})}$

$\arg(zw) = \theta + \frac{2\pi}{3}$

$\tan(\arg(zw)) = \tan(\theta+\frac{2\pi}{3}) = \dfrac{\tan\theta+\tan\frac{2\pi}{3}}{1-\tan\theta\tan\frac{2\pi}{3}}$

$= \dfrac{\tan\theta-\sqrt{3}}{1-\sqrt{3}\tan\theta}$

$= \dfrac{\frac{1}{3}-\sqrt{3}}{1-\sqrt{3}(\frac{1}{3})}$

$= \dfrac{-3+4\sqrt{3}}{3}$

$\therefore \tan^2(\arg(zw)) = \dfrac{19+8\sqrt{3}}{3}$

$\sec^2(\arg(zw)) = \dfrac{22+8\sqrt{3}}{3}$

$\cos(\arg(zw)) = \sqrt{\dfrac{3}{22+8\sqrt{3}}}$

$\therefore \sin(\arg(zw))$

B1

A is $6+2i$

B is at $2\sqrt{10}\,e^{i(\theta+\frac{2\pi}{3})} = 2\sqrt{10}\,e^{i(\arctan(\frac{1}{3})+\frac{2\pi}{3})}$

$\therefore$ B is at $2\sqrt{10}\left(\cos\left(\frac{2\pi}{3}+\arctan(\frac{1}{3})\right)+i\sin\left(\frac{2\pi}{3}+\arctan(\frac{1}{3})\right)\right)$

B is at $-4.732+4.196i$

C is at $2\sqrt{10}e^{i\theta}\left(e^{-\frac{2i\pi}{3}}\right) = 2\sqrt{10}\,e^{i(\theta-\frac{2\pi}{3})}$

$\therefore$ C is at $-1.268 - 6.196i$

2/9

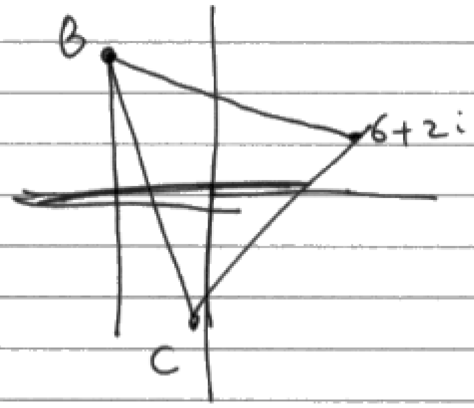
Examiner Comments: (a) M1A0A0M1A0A0 (b) M0M0A0

In part (a), this candidate adopts a correct strategy for finding the points *B* and *C* by multiplying the exponential form of 6 + 2i by $e^{\frac{2\pi}{3}}$ and $e^{-\frac{2\pi}{3}}$ but does not obtain any of the required values in the required exact form.

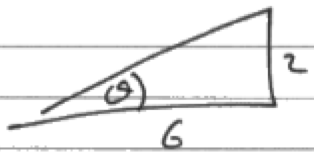
There is no attempt at part (b).

Student Response B

a)



~~6+2i =~~



$\tan(\theta) = \frac{2}{6} = \frac{1}{3}$

$\theta = \arctan\left(\frac{1}{3}\right)$

$r = 2\sqrt{10}$

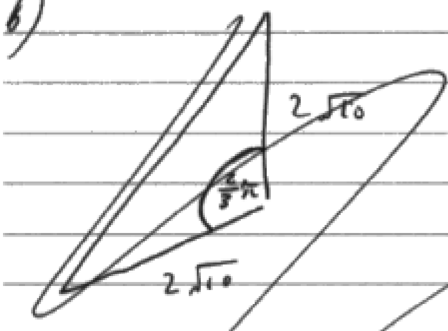
$6 + 2i = 2\sqrt{10}\,e^{\arctan\left(\frac{1}{3}\right)i}$

$B = 2\sqrt{10}\,e^{\arctan\left(\frac{1}{3}\right)i} \times e^{\frac{2}{3}\pi i} =$

$2\sqrt{10}\,e^{\left(\arctan\left(\frac{1}{3}\right) + \frac{2}{3}\pi\right)i}$

$C = 2\sqrt{10}\,e^{\left(\arctan\left(\frac{1}{3}\right) + \frac{4}{3}\pi\right)i}$

b)

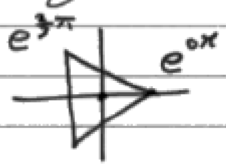


$\frac{1}{2}ab\sin C = \frac{1}{2}(2\sqrt{10})^2 \sin\left(\frac{2}{3}\pi\right)$

$= 10\sqrt{3}$

∴ whole triangle

$3 \times 10\sqrt{3} = 30\sqrt{3}$



$A = (1, 0)$ & $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = B$

for other triangle

midpoint of A & B $= \left(\frac{1}{4} + \frac{\sqrt{3}}{4}i\right)$

$= \frac{1}{2}e^{\frac{1}{3}\pi i}$

∴ ~~length of~~ distance from vertices halves at midpoint.

5/9

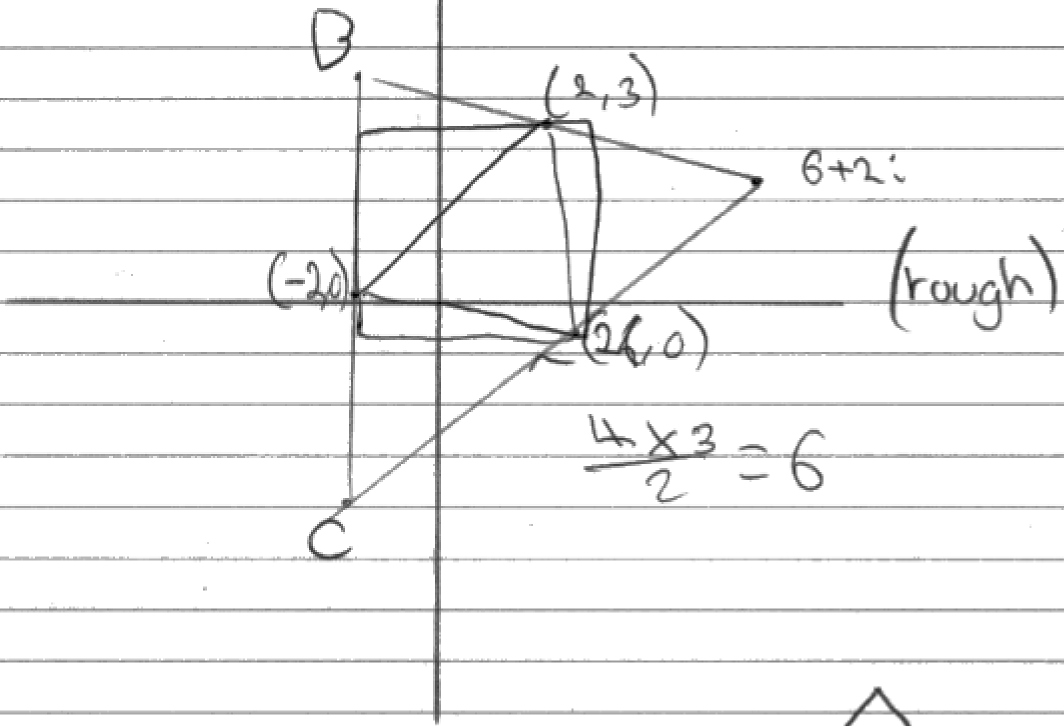
Examiner Comments: (a) M1A0A0M1A0A0 (b) M1M1A1

In part (a), this candidate adopts a correct strategy for finding the points *B* and *C* by multiplying the exponential form of 6 + 2i by $e^{\frac{2\pi}{3}}$ and $e^{-\frac{2\pi}{3}}$ but does not obtain any of the required values in the required exact form.

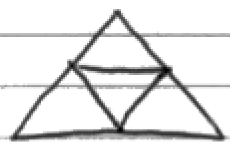
In part (b), this candidate correctly deduces that the distance from the origin to a vertex of triangle *DEF* is half the modulus of 6 + 2i and uses this correctly to find the area of triangle *DEF*.

Student Response C

a.



$\frac{4\times 3}{2} = 6$



$z^3 = 1$

$|z| = 1$

$\arg z^3 = 0$

$\arg z = \frac{0}{3} + \frac{2\pi k}{3}$

$k=0 \quad \arg z = 0$

$k=1 \quad \arg z = \frac{2\pi}{3}$

$k=2 \quad \arg z = \frac{4\pi}{3}$

~~$\arg(6+2i) = \tan^{-1}\left(\frac{2}{6}\right)$~~

$k=0: \quad z_1 = 1 + 0i$

$k=1: \quad z = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) =$

$z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$k=2: \quad z = \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$= -\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)$

$z_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$z_2(6+2i) = (-3-\sqrt{3}) + i(-1+3\sqrt{3})$ (B)

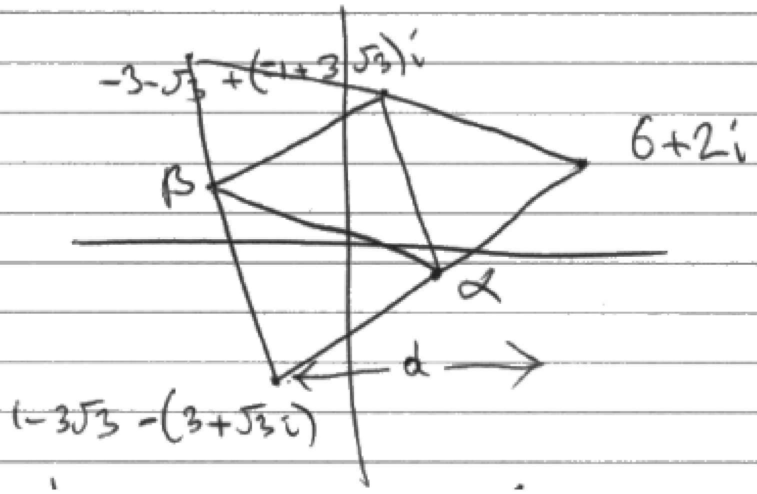
$z_3(6+2i) = (1-3\sqrt{3}) - (3+\sqrt{3})i$ (C)

$z_1 = 6+2i$

b. $\frac{1}{4} \times \sqrt{6^2+2^2} \times \left(\frac{1}{2} \times 1 \times \sin 60\right)$

as its $\frac{1}{4}$ of larger triangle

$= \frac{5\sqrt{3}}{2}$



$\frac{1}{2}(6+3\sqrt{3}-1)(1+3\sqrt{3}+(2+\sqrt{3}))$

$\alpha\left(\frac{7-3\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}\right)$

$\beta\left(\frac{-2-4\sqrt{3}}{2}, \frac{-4+2\sqrt{3}}{2}\right)$

length

$\sqrt{\frac{1}{4}(7+2+\sqrt{3})^2+\frac{1}{4}(3+3\sqrt{3})^2} = C$

$\frac{1}{2}C^2 \times \sin 60 = 19.74$

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Examiner Comments: (a) M1A1A1M1A1A1 (b) M1M1A1

In part (a), this candidate adopts a correct strategy for finding the points *B* and *C* by multiplying the exponential form of 6 + 2i by $e^{\frac{2\pi}{3}}$ and $e^{-\frac{2\pi}{3}}$ and then obtains the two complex numbers in the required exact form.

In part (b), this candidate correctly finds the length of one side of triangle *ABC* and then uses this to deduce the length of one side of triangle *DEF* and then applies $\frac{1}{2}ab\sin C$ to obtain the correct area.

Exemplar Question 7

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7

$$\mathbf{M} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(*a*) Find the values of k for which the matrix **M** has an inverse.

(2)

(*b*) Find, in terms of p, the coordinates of the point where the following planes intersect

$$2x - y + z = p$$
$$3x - 6y + 4z = 1$$
$$3x + 2y - z = 0$$

(5)

(*c*) (i) Find the value of q for which the set of simultaneous equations

$$2x - y + z = 1$$
$$3x - 5y + 4z = q$$
$$3x + 2y - z = 0$$

can be solved.

(ii) For this value of q, interpret the solution of the set of simultaneous equations geometrically.

(4)

(Total for Question 7 is 11 marks)

Mean Score 7.0 out of 11

Examiner Comments

This matrix question also saw a wide range in the quality of response. In part (a) it was surprising to see a significant number of slips in obtaining an expression for the determinant. Most obtained an expression for det **M** conventionally although a few used the rule of Sarrus – usually correctly. A small number gave their answer as "$k = 5$" but the question required the values of k for which **M** had an inverse and not the value of k for which **M** was singular.

Part (b) required the point of intersection of three planes to be found and the most successful students used Way 1. It is acceptable to obtain an inverse of a matrix with no variables as elements using a calculator and it was unfortunate to see some embarking upon a step-by-step method. This often led to errors such as omitting the $\frac{1}{\det \mathbf{M}}$ multiplier. The correct inverse was seen fairly widely and the subsequent matrix multiplication was also often correct. This specification has an assessment objective for the use of correct notation so the point of intersection had to be given as coordinates. Those who chose to solve the system of equations were much less successful, with many unable to obtain x, y and z in terms of p (including a small number who attempted to find a value for p).

Those who did not make any progress in (b) often left (c) unanswered but this part was still a reasonable source of marks for many. In part (i), correct strategies to obtain a value of q were common and although slips were evident, the correct $q = 3$ was often achieved. Weaker attempts tried to use an inverse, even with students who had scored both marks in (a). Part (ii) required a geometric interpretation of the solution to the equations. Some neglected to mention "planes". A few students did this successfully with a diagram.

Mark Scheme

Question	Scheme	Marks	AOs
7(a)	$\lvert\mathbf{M}\rvert = 2(-k-8)+1(-3-12)+1(6-3k)=0 \Rightarrow k = \ldots$	M1	1.1b
	$k \neq {}^{-}5$	A1	2.4
		(2)	
(b) Way 1	$\mathbf{M}=\begin{pmatrix}2&-1&1\\3&-6&4\\3&2&-1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}p\\1\\0\end{pmatrix}$	M1	3.1a
	$\mathbf{M}^{-1}=\frac{1}{5}\begin{pmatrix}-2&1&2\\15&-5&-5\\24&-7&-9\end{pmatrix}$	B1	1.1b
	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-2&1&2\\15&-5&-5\\24&-7&-9\end{pmatrix}\begin{pmatrix}p\\1\\0\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\ldots$	M1	2.1
	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-2p+1\\15p-5\\24p-7\end{pmatrix}$	A1	1.1b
	$\left(\frac{-2p+1}{5},\ 3p-1, \frac{24p-7}{5}\right)$	A1ft	2.5
		(5)	
(b) Way 2	$2x-y+z=p$ $3x-6y+4z=1 \qquad 8y-5z=-1$ $3x+2y-z=0 \Rightarrow \text{e.g. } 9y-5z=3p-2 \Rightarrow y=\ldots$ $\Rightarrow x=\ldots, z=\ldots$	M1	3.1a
	$y=3p-1$ (or $x=\frac{-2p+1}{5}$ or $z=\frac{24p-7}{5}$)	B1	1.1b
	$8(3p-1)-5z=-1 \Rightarrow z=\ldots \Rightarrow x=\ldots$	M1	2.1
	$z=\frac{24p-7}{5}$, $x=\frac{-2p+1}{5}$	A1	1.1b
	$\left(\frac{-2p+1}{5},\ 3p-1, \frac{24p-7}{5}\right)$	A1ft	2.5

	For consistency:	M1	3.1a

(c)(i)	E.g. $5x+y=4-q$ and $15x+3y=q$		
	$4-q=\frac{q}{3} \Rightarrow q=...$	M1	2.1
	$q=3$	A1	1.1b
	Alternative for (c)(i): $x=1 \Rightarrow 2-y+z=1,\ 3+2y-z=0 \Rightarrow y=...,z=...$ M1 for allocating a number to one variable and solves for the other 2 $x=1, y=-4, z=-5 \Rightarrow 3+20-20=q$ M1 substitutes into the second equation and solves for q A1: $q=3$		
(ii)	Three **planes** that intersect in a **line** Or Three **planes** that form a **sheaf** allow **sheath**!	B1	2.4
		(4)	
			(11 marks)

Notes

(a)
M1: Attempts determinant, equates to zero and attempts to solve for k in order to establish the restriction for k. For the determinant, at least 2 of the 3 "elements" should be correct.
May see rule of Sarrus used for determinant e.g.
$|\mathbf{M}|=(2)(k)(-1)+(4)(3)(-1)+(3)(2)(1)-(3)(k)(-1)-(2)(4)(2)-(-1)(3)(-1)=0 \Rightarrow k=...$
A1: Describes the correct condition for k with no contradictions. Allow e.g. $k < -5$, $k > -5$
(b)**Way 1**
M1: A complete strategy for solving the given equations. Need to see an attempt at the inverse followed by a correct method for finding x, y and z
B1: Correct inverse matrix
M1: Uses their inverse and attempts the multiplication with the correct vector
A1: Correct values for x, y and z in any form
A1ft: Correct values given in coordinate form only. **Follow through their *x*, *y* and *z*.**
Way 2
M1: A complete strategy for solving the given equations. Need to see an attempt at eliminating one variable followed by a correct method for finding x, y and z
B1: One correct value
M1: Uses the equations to find values for the other 2 variables
A1: Correct values for x, y and z in any form
A1ft: Correct values given in coordinate form only. **Follow through their *x*, *y* and *z*.**
(c)(i)
M1: Uses a correct strategy that will lead to establishing a value for q. E.g. eliminating one of x, y or z
M1: Solves a suitable equation to obtain a value for q
A1: Correct value
(ii)
B1: Describes the correct geometrical configuration.
Must include the **two** ideas of **planes** and meeting in a **line** or forming a **sheaf** with no contradictory statements.

Student Response A

7a) For inverse det $M \neq 0$

$M = \begin{pmatrix} 2 & -1 & 1 \\ 3 & k & 4 \\ 3 & 2 & -1 \end{pmatrix}$

Det $M = 2\begin{vmatrix} k & 4 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & k \\ 3 & 2 \end{vmatrix}$

$= 2(-k+8) + 1(-3+12) + 1(6+3k)$ det $M = 25$

$= -2k + 16 - 3 + 12 + 6 + 3k \neq 0$

$k + 31 \neq 0$

$k \neq -31$

k is any real number other than -31

b) $2x - y + z = p$

$3x - 6y + 4z = 1$

$3x + 2y - z = 0$

$3x = -2y + z$ and $3x = 1 + 6y - 4z$

$-2y + z = 1 + 6y - 4z$ $\quad x = \frac{1 + 6y - 4z}{3}$

$-2y - 6y = 1 - 4z - z$

$-8y = 1 - 5z$

$-8y + 5z = 1$

$-8y = 1 - 5z$

$-y = \frac{1 - 5z}{8}$ $\quad y = \frac{5z - 1}{+8}$ ①

$-\frac{1}{2}$

$$2x - \left(\frac{1-5z}{8}\right) + z = p$$

$$2x \frac{-1+5z}{8} + z = p$$

$$16x - 1 + 5z + 8z = 8p$$

$$16\left(\frac{1+6y-4z}{3}\right) - 1 + 5z + 8z = 8p$$

$$\frac{16+96y-64z}{3} - 1 + 13z = 8p \quad \text{Sub ① in}$$

$$\frac{16+\overset{12}{96}\left(\frac{5z-1}{8}\right) - 64z}{3} - 1 + 13z = 8p$$

$$\frac{16 + 12(5z-1) - 64z}{3} + 13z - 1 = 8p$$

✓ ✓ ✓

$$\frac{16 + 60z - 12 - 64z}{3} + \frac{39z - 3}{3} = 8p$$

$$1 + 35z = 24p$$

$$35z = 24p - 1$$

$$\boxed{z = \frac{24p - 1}{35}}$$

$$y = \frac{8\left(\frac{24p-1}{35}\right) - 1}{8} = \frac{168p - 7 - 1}{8}$$

$$= \frac{168p - 8}{8}$$

$$\boxed{y = 21p - 1}$$

$$x = \frac{1 + 6y - 4z}{3}$$

$$= \frac{1 + 6(2p - 1) - 4\left(\frac{24p - 1}{35}\right)}{3}$$

$$= 1 + 12p - 6 - \frac{96p + 4}{35}$$

$$= \frac{35 + 4410p - 210 - 96p + 4}{35}$$

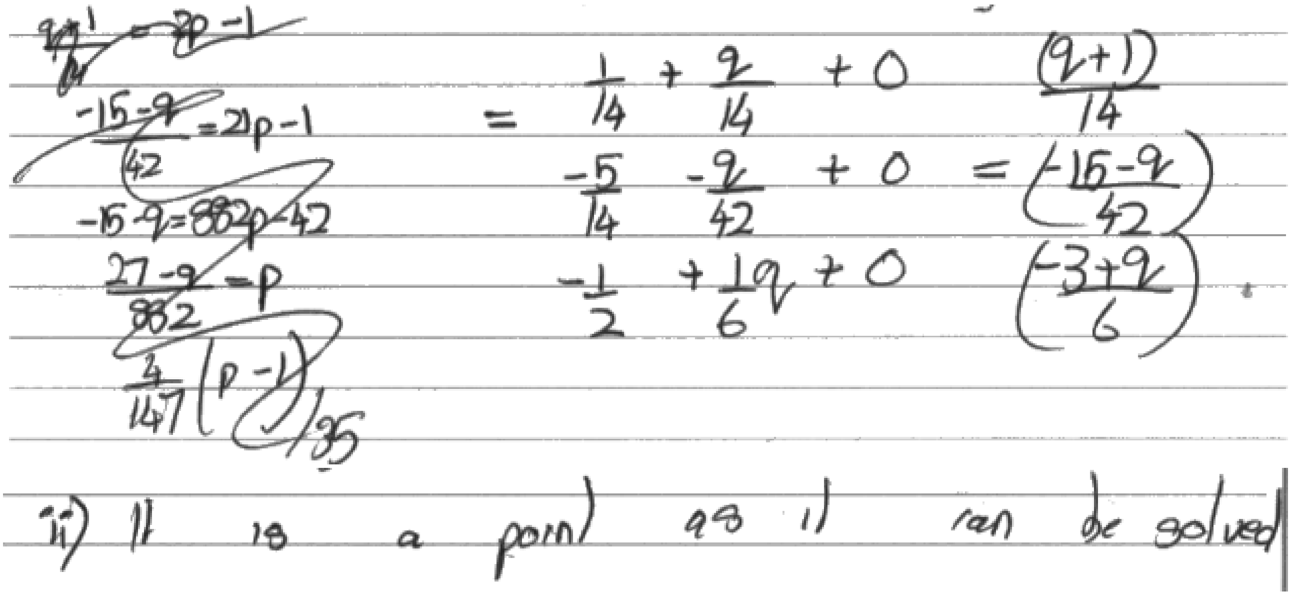
$$\boxed{\frac{-171 + 4314p}{35} = x}$$

c. i) ~~2x − 1~~

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 4 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 4 \\ 3 & 2 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 9 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & \frac{1}{14} & \frac{3}{14} \\ \frac{-5}{14} & \frac{-1}{42} & \frac{11}{42} \\ \frac{-1}{2} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 1 \\ 9 \\ 0 \end{pmatrix}$$



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Examiner Comments: (a) M0A0 (b) M1B0M1A0A0 (c) M0M0A0B0

In part (a), the attempt at the determinant is incorrect (at least two correct "elements" were required to score the method mark).

In part (b), this candidate scored both method marks for using elimination to solve the equations resulting in expressions in p for all three variables. As their answers were not given as coordinates, the follow through mark was unavailable.

In part (c)(i), the candidate incorrectly attempts to solve the problem using an inverse matrix and so no marks are scored.

The description in (c)(ii) is incorrect.

Student Response B

(4)

a) det $M \neq 0$

$2(-k-8)$
$-(-3-12)$
$+(6-3k)$

$-2k-16+3+12+6-3k=0$

$5=5k$

$k \neq 1, \ k \in \mathbb{R}$

b)

$6x = 3y - 3z + 3p$
$6x = 12y - 8z + 2$
$6x = -4y + 2z$

$3y - 3z + 3p = 12y - 8z + 2 = -4y + 2z$

$-4y$

~~3y – 3z~~ $3z - 3y = 3p$

$8z - 12y = 2$

$12z - 12y = 12p$

$4z = 12p - 2$

$z = 3p - 1/2$

$9p - 3/2 - 3p = 3y$

$3y = 6p - 3/2$

$y = 2p - 1/2$

$6x = 24p - 6 - 24p + 4 + 2$

$6x = 0$

$$\begin{pmatrix} 0 \\ 2p - 1/2 \\ 3p - 1/2 \end{pmatrix}$$

c) i)

$2x - y + z = 1$

$3x - 5y + 4z = q$

$8x - 4y + 4z = 1$

~~12x – 5y~~

$5x + y = 1 - q$

$q = 0$

$2x - y + z = 1$

$3x + 2y - z = 0$

$5x + y = 1$

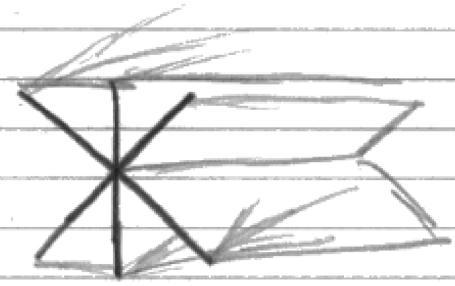
ii) $3x - 5y + 4z = 0$

$3x + 2y - z = 0$

$12x + 8y - 4z$

$15x + 3y = 0$ $\quad 3x \quad 5x + y = 0$

all meet along one line $\Rightarrow$ sheaf



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Examiner Comments: (a) M1A0 (b) M1B0M1A0A0 (c) M1M1A0B1

In part (a), the attempt at the determinant is acceptable as at least two "elements" are correct so this part scores M1A0 as the candidate sets their determinant = 0 and solves for k.

In part (b), this candidate scored both method marks for using elimination to solve the equations resulting in expressions in p for all three variables. As their answers were not given as coordinates, the follow through mark was unavailable.

In part (c)(i), the candidate eliminates one variable and obtains a value for q and so both method marks are scored.

The description in (c)(ii) was allowed for the "sheaf" and the idea of "planes" was implied from their diagram.

Student Response C

a) For M^{-1} to exist $\det M \neq 0$

$\det M = 2\begin{vmatrix} k & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 3 & k \\ 3 & 2 \end{vmatrix} = 2(-k-8) + (-3-12) + (6-3k)$

$= -2k - 16 - 3 - 12 + 6 - 3k = -5k - 25$

$-5k - 25 = 0 \rightarrow 5k = -25 \rightarrow k = -5$

b) $\begin{pmatrix} 2 & -1 & 1 \\ 3 & -6 & 4 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ 3 & -1 & -1 \\ \frac{24}{5} & -\frac{7}{5} & -\frac{9}{5} \end{pmatrix}\begin{pmatrix} p \\ 1 \\ 0 \end{pmatrix} \rightarrow$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{2}{5}p + \frac{1}{5} \\ 3p - 1 \\ \frac{24}{5}p - \frac{7}{5} \end{pmatrix}$

SO coordinates of the planes intersection is $\left(-\frac{2}{5}p + \frac{1}{5}, 3p-1, \frac{24}{5}p - \frac{7}{5}\right)$

c) i) $\begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 4 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ q \\ 0 \end{pmatrix}$ ~~$\rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}$~~ $= \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 4 \\ 3 & 2 & -1 \end{pmatrix}$ has no inverse as the det $= 0$

$2x - y + z = 1$ ①

$3x - 5y + 4z = q$ ②

$3x + 2y - z = 0$ ③

③ − ②: $7y - 5z = -q$

① × 3: $6x - 3y + 3z = 3$ Ⓐ

② × 2: $6x - 10y + 8z = 2q$ Ⓑ

Ⓐ − Ⓑ: $7y - 5z = 3 - 2q$

$3 - 2q = -q \rightarrow q = 3$

ii) System of equations is consistent and has infinitely many solutions, the planes form a ~~pr~~ prism

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Examiner Comments: (a) M1A0 (b) M1B1M1A1A1 (c) M1M1A1B0

In part (a), the attempt at the determinant is correct and the candidate sets their determinant = 0 and solves for k so the method mark is scored. The conclusion is incorrect and so the accuracy mark is not scored.

In part (b), this candidate uses a fully correct inverse matrix method to solve the system of equations. The candidate also gives their answer in coordinate form and so scores full marks in this part.

In part (c)(i), the candidate eliminates one variable and proceeds to obtain the correct value for *q* and so full marks are scored.

The description in (c)(ii) is incorrect.

Exemplar Question 8

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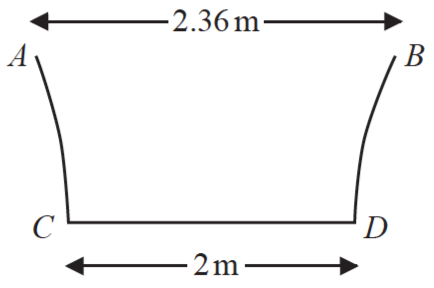

Figure 1

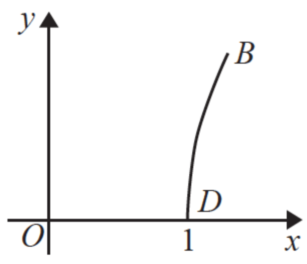

Figure 2

Figure 1 shows the central vertical cross section *ABCD* of a paddling pool that has a circular horizontal cross section. Measurements of the diameters of the top and bottom of the paddling pool have been taken in order to estimate the volume of water that the paddling pool can contain.

Using these measurements, the curve *BD* is modelled by the equation

$$y = \ln(3.6x - k) \qquad 1 \leqslant x \leqslant 1.18$$

as shown in Figure 2.

(*a*) Find the value of *k*.

(1)

(*b*) Find the depth of the paddling pool according to this model.

(2)

The pool is being filled with water from a tap.

(*c*) Find, in terms of *h*, the volume of water in the pool when the pool is filled to a depth of *h* m.

(5)

Given that the pool is being filled at a constant rate of 15 litres every minute,

(*d*) find, in cm h^{-1}, the rate at which the water level is rising in the pool when the depth of the water is 0.2 m.

(3)

(Total for Question 8 is 11 marks)

Mean Score 6.3 out of 11

Examiner Comments

The last question challenged many, but there were a lot of accessible marks here, although a fully correct solution to part (d) was rarely seen.

In part (a), most were able to use the model with $x = 1$ and $y = 0$ to find k correctly. Students were slightly less successful in part (b) however, with a few unable to recognise the need to use the 1.18 given beside the model equation for curve *BD*. A small number left their answer as a natural logarithm.

Most knew that a volume of revolution was required in part (c) although some omitted the π from the formula (or had 2π) or they attempted $\int y^2 \mathrm{d}x$ rather than $\int x^2 \mathrm{d}y$. Most made x the subject of the formula correctly although slips were seen in squaring, including failing to find a middle term or not squaring the denominator in their single fraction expression for x. Integration was commonly successful although a significant number neglected to substitute the zero limit. Use of the answer to part (b) as the upper limit was occasionally seen.

Part (d) proved discriminating although most who made an attempt recognised that the chain rule could be deployed and for the most part it was used correctly. Weaker attempts tended to involve calculating V and then attempting to adjust its value. Many otherwise successful students were unable to correctly manage the different units used in the question. A very small number of exceptional candidates were able to deduce that the rate of change of h with respect to time was proportional to the circular surface area of the pool and correctly proceeded without any need for calculus.

Mark Scheme

Question	Scheme	Marks	AOs
8(a)	$k = 2.6$	B1	3.4
		(1)	
(b)	$x = 1.18 \Rightarrow \ln(3.6\times1.18 - "2.6") = \ldots$	M1	1.1b
	$h = 0.4995\ldots$ m	A1	2.2b
		(2)	
(c)	$y = \ln(3.6x - 2.6) \Rightarrow x = \dfrac{e^y + 2.6}{3.6}$ or $\dfrac{5e^y + 13}{18}$	B1ft	1.1a
	$V = \pi\int\left(\dfrac{e^y + 2.6}{3.6}\right)^2 dy = \dfrac{\pi}{3.6^2}\int\left(e^{2y} + 5.2e^y + 6.76\right)dy$ or $\dfrac{\pi}{324}\int\left(25e^{2y} + 130e^y + 169\right)dy$	M1	3.3
	$= \dfrac{\pi}{3.6^2}\left[\dfrac{1}{2}e^{2y} + 5.2e^y + 6.76y\right]$ $\left(\text{or } \dfrac{\pi}{324}\left[\dfrac{25}{2}e^{2y} + 130e^y + 169y\right]\right)$	A1	1.1b
	$= \dfrac{\pi}{3.6^2}\left\{\left(\dfrac{1}{2}e^{2h} + 5.2e^h + 6.76h\right) - \left(\dfrac{1}{2}e^0 + 5.2e^0 + 6.76(0)\right)\right\}$ or e.g. $= \dfrac{\pi}{324}\left\{\left(\dfrac{25}{2}e^{2h} + 130e^h + 169h\right) - \left(\dfrac{25}{2}e^0 + 130e^0 + 6.76(0)\right)\right\}$	M1	2.1
	$= \dfrac{\pi}{3.6^2}\left(\dfrac{1}{2}e^{2h} + 5.2e^h + 6.76h - 5.7\right)$	A1	1.1b
		(5)	
(d)	$\dfrac{dV}{dh} = \dfrac{\pi}{3.6^2}\left(e^{2h} + 5.2e^h + 6.76\right) = \dfrac{\pi}{3.6^2}\left(e^{0.4} + 5.2e^{0.2} + 6.76\right)$	M1	3.1a
	$\dfrac{dh}{dt} = \dfrac{dh}{dV}\dfrac{dV}{dt} = \dfrac{1}{3.539\ldots}\times 0.015\times 60$	M1	1.1b
	$\dfrac{dh}{dt} = 25.4\,\text{cm}\,\text{h}^{-1}$	A1	3.2a
		(3)	
(d) Way 2	$y = 0.2 \Rightarrow x = \dfrac{2.6 + e^{0.2}}{3.6} \Rightarrow A = \pi\left(\dfrac{2.6 + e^{0.2}}{3.6}\right)^2 (= 3.54)$	M1	3.1a
	$\dfrac{dh}{dt} = \dfrac{0.015\times 60}{3.54}$	M1	1.1b
	$\dfrac{dh}{dt} = 25.4\,\text{cm}\,\text{h}^{-1}$	A1	3.2a
			(11 marks)

Notes
(a) B1: Uses the model to obtain a correct value for k. Must be 2.6 not -2.6 (b) M1: Substitutes their value of k and x = 1.18 into the given model to find a value for y A1: Infers that the depth of the pool could be awrt 0.5 m (c) B1ft: Uses the model to obtain x correctly in terms of y (follow through their k) M1: Uses the model to obtain an expression for the volume of the pool using $\pi\int(their\ f(y))^2\,\mathrm{d}y$ – must expand in order to reach an integrable form (allow poor squaring e.g. $(a + b)^2 = a^2 + b^2$. **Note that the π may be recovered later.** A1: Correct integration M1: Selects limits appropriate to the model (h and 0) substitutes and clearly shows the use of both limits (i.e. including zero) A1: Correct expression **(allow unsimplified and isw if necessary)** (d) **Way 1** M1: Recognises that $\frac{\mathrm{d}V}{\mathrm{d}h}$ is required and attempts to find $\frac{\mathrm{d}V}{\mathrm{d}h}$ or $\frac{\mathrm{d}h}{\mathrm{d}V}$ from their integration or using the earlier result (before integrating). Must clearly be identified as $\frac{\mathrm{d}V}{\mathrm{d}h}$ or $\frac{\mathrm{d}h}{\mathrm{d}V}$ unless this implied by subsequent work. M1: Evidence of the correct use of the chain rule (ignore any confusion with units). Look for an attempt to divide 15 or their converted 15 by their $\frac{\mathrm{d}V}{\mathrm{d}h}$ or to multiply 15 or their converted 15 by $\frac{\mathrm{d}h}{\mathrm{d}V}$ **but must reach a <u>value</u> for** $\frac{\mathrm{d}h}{\mathrm{d}t}$ **<u>but you do not need to check their value.</u>** A1: Interprets their solution correctly to obtain the correct answer (awrt 25.4) **with the correct units** **Way 2** M1: Uses y = 0.2 to find x and the surface area of the water at that instant M1: Attempts to divide the rate by their area (ignore any confusion with units) A1: Interprets their solution correctly to obtain the correct answer (awrt 25.4) **with the correct units**

Student Response A

a) $0 = \ln(3.6 - k)$

$e^0 = 3.6 - k$

$k = 2.6$

b) $x = 1.18$

$y = \ln(3.6 \times 1.8 + 2.6)$

$= \sout{2.20607}\ldots \quad 1.3558\ldots$

$= \sout{2.21m} \quad 1.36m$

c) $\text{vol} = \pi \int_0^h x^2 \, dy$

$y = \ln(3.6x - 2.6)$

$e^y = 3.6x - 2.6$

$3.6x = e^y + 2.6$

$x = \frac{5}{18}e^y + \frac{13}{18}$

$x^2 = \left(\frac{5}{18}e^y + \frac{13}{18}\right)^2$

$= \frac{25}{324}e^{2y} + \frac{65}{162}e^y + \frac{169}{324}$

$\text{volume} = \pi \int \frac{25}{324}e^{2y} + \frac{65}{162}e^y + \frac{169}{324} \, dy$

$= \pi \times \frac{1}{324} \int 25e^{2y} + 130e^y + 169 \, dy$

$= \frac{\pi}{324}\left[\frac{25}{2}e^{2y} + 130e^y + 169y\right]_0^h$

$= \frac{25\pi}{648}e^{2h} + \frac{130\pi}{324}e^h + \frac{169\pi}{324}h$

$= \frac{25\pi}{648}e^{2h} + \frac{65\pi}{162}e^h + \frac{169\pi}{324}h$

d) $\frac{dV}{dt} = 15$ Litres per minute

$= 0 \cdot 25$ litres per hour

Volume when $h = 0.2$ is $1.8247\ldots$ litres

rate $= \frac{dV}{dt} = \frac{25\pi}{324} e^{2h} + \frac{65\pi}{81} e^{h} + \frac{169\pi}{324}$

let $e^h = k \rightarrow e^h = 3.8641..$

$h = 1.3517..$

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Examiner Comments: (a) B1 (b) M0A0 (c) B1M1A1M0A1 (d) M0M0A0

In part (a) the B1 is scored for $k = 2.6$

In part (b) the method mark is not scored as the candidate uses an incorrect equation e.g. $y = \ln(3.6 + k)$ rather than $y = \ln(3.6 - k)$

In part (c), this candidate correctly finds x in terms of y, squares and applies the correct volume formula and integrates correctly and thus scores the first 3 marks. When applying limits, the lower limit of 0 is not considered and so the final 2 marks are not scored.

There is no creditable work in part (d).

Student Response B

a) ~~y = ln(3.6x + h)~~

when $x = 1$, $y = 0$

$0 = \ln(-h)$ $\quad h = -1$

b) $y = \ln(3.6 \times 1.18 + 1)$

$= 1.66\,m$

c) $y = \ln(3.6x + 1)$

$e^y = 3.6x + 1$

$x = \dfrac{e^y - 1}{3.6}$

$x^2 = \dfrac{e^{2y} - 2e^y + 1}{12.96}$

$V = \pi \displaystyle\int_0^h \frac{e^{2y} - 2e^y + 1}{12.96}\,dy$

$= \pi \left[\dfrac{e^{2y}}{25.92} - \dfrac{e^y}{6.48} + \dfrac{y}{12.96} \right]_0^h$

$= \pi\left(\dfrac{25e^{2h}}{648} - \dfrac{25e^h}{162} + \dfrac{25h}{324}\right) - \pi\left(\dfrac{25}{648} - \dfrac{25}{162}\right)$

d) $\dfrac{dV}{dt} = 2.5 \times 10^{-4}$ $\qquad 15\,l/min = 900\,l/hour$

$= 0.9\,m^3/hour$

$2.5 \times 10^{-4}\,m^3/hour$

$\dfrac{dV}{dy}$ $\qquad \dfrac{dV}{dh} = \pi\left(\dfrac{25e^{2h}}{324} - \dfrac{25e^h}{162} + \dfrac{25}{324}\right)$

$\dfrac{dV}{dh}$ $\quad \dfrac{1}{dt}$ $\quad = 0.01188$

$\dfrac{dV}{dh} = \dfrac{dl}{dh}$ $\qquad \dfrac{dh}{dt} = \dfrac{2.5 \times 10^{-4}}{0.01188} = 0.0210\,mh^{-1}$

$= 2.1\,cmh^{-1}$

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Examiner Comments: (a) B0 (b) M1A0 (c) B1M1A0M1A0 (d) M1M1A0

In part (a), $k = -1$ is incorrect

In part (b), the candidate uses their value of k correctly to find y and scores the method mark but the accuracy mark is not scored due to the incorrect value of k.

In part (c), uses a fully correct approach and scores the follow through B mark as well as the method marks.

In part (d), the candidate also applies a correct method for the required rate and scores both method marks.

Student Response C

a. $x=1,\ y=0$

$0 = \ln(3.6 - k)$

$1 = 3.6 - k \qquad k = 2.6$

b. $x = 1.18$

$y = \ln(3.6(1.18) - 2.6)$

$= 0.4996\text{ m}$

$= 0.500\text{m deep}$

c. $\pi \int x^2 \, dy$

$e^y = 3.6x - 2.6$

$2.6 + e^y = 3.6x$

$x = \dfrac{2.6 + e^y}{3.6}$

$x^2 = \dfrac{1}{3.6^2}\left(2.6^2 + 5.2e^y + e^{2y}\right)$

$\dfrac{\pi}{3.6^2}\int_0^h 6.76 + 5.2e^y + e^{2y} \, dy$

$= \dfrac{\pi}{3.6^2}\left[6.76y + 5.2e^y + \frac{1}{2}e^{2y}\right]_0^h$

$= \dfrac{\pi}{3.6^2}\left(\left(6.76h + 5.2e^h + \frac{1}{2}e^{2h}\right) - \left(0 + 5.2 + \frac{1}{2}\right)\right)$

$= \dfrac{\pi}{3.6^2}\left(6.76h + 5.2e^h + \frac{1}{2}e^{2h} - 5.7\right)$

d. $0.2m = y \quad x = \frac{2.6 + e^{0.2}}{3.6}$

$= 1.062m$

CSA: $\pi x^2 = 3.54m^2$

15 Litre = $0.015m^3/min = 0.9m^3/hour$

$\frac{0.9m^3/hour}{3.54m^2} = 0.254m/hour$

$= 25.4cm/hour.$

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Examiner Comments: (a) B1 (b) M1A1 (c) B1M1A1M1A1 (d) M1M1A1

This candidate has a fully correct response in all parts of the question.